Central limit theorem for products of toral automorphisms

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Abstract

Let (τ_n) be a sequence of toral automorphisms $\tau_n : x \to A_n x \bmod \mathbb{Z}^d$ with $A_n \in \mathcal{A}$, where $\mathcal{A}$ is a finite set of matrices in $SL(d, \mathbb{Z})$. Under some conditions the method of "multiplicative systems" of Komlòs can be used to prove a Central Limit Theorem for the sums $\sum_{k=1}^n f(\tau_k \circ \tau_{k-1} \cdots \circ \tau_1 x)$ if f is a Hölder function on $\mathbb{T}^d$. These conditions hold for 2×2 matrices with positive coefficients. In dimension d they can be applied when $A_n = A_n(\omega)$, with independent choices of $A_n(\omega)$ in a finite set of matrices $\in SL(d, \mathbb{Z})$, in order to prove a "quenched" CLT.

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Introduction

Let us consider a sequence of maps obtained by composition of transformations (τ_n) acting on a probability space $(X, \mathcal{B}, \lambda)$. The iteration of a single measure preserving transformation corresponds to the classical case of a dynamical system. The case of several transformations has been also considered by some authors, and the stochastic behavior of the sums $\sum_{k=1}^n f(\tau_k \circ \tau_{k-1} \cdots \circ \tau_1 x)$, for a function f on X, has been studied on some examples. For example the notion of stochastic stability is defined using composition of transformations chosen at random in the neighborhood of a given one. Bakhtin considered in [3] non perturbative cases with geometrical assumptions on the transformations. In the non-invertible case, the example of sequences of expanding maps of the interval was carried out in [4].

Here we consider the example of automorphisms of the torus. Given a finite set $\mathcal{A}$ of matrices in $SL(d, \mathbb{Z})$, to a sequence $(A_i)_{i \in \mathbb{N}}$ taking values in $\mathcal{A}$ corresponds the sequence $(\tau_i)_{i \in \mathbb{N}}$ of automorphisms of the torus $\mathbb{T}^d$ defined by: $\tau_i : x \mapsto A_i^t x$ mod 1. If the choice in $\mathcal{A}$ of the matrices is random, we write $A_i(\omega)$ and $\tau_i(\omega)$.

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Let $f : \mathbb{T}^d \to \mathbb{R}$ be a Hölder function with integral zero. A question is the existence of the variance and the central limit theorem for the sums $S_N f = \sum_{k=1}^N f(\tau_k ... \tau_1 .)$ and the Lebesgue measure λ on the torus.

When the matrices are chosen at random and independently, our problem is strongly related to the properties of a random walk on $SL(d, \mathbb{Z})$. Among the many works on random walks on groups let us mention a paper of Furman and Shalom which deals with questions directly connected to ours. Let μ be a probability measure on $SL(d, \mathbb{Z})$. Let $\mathbb{P} = \mu^{\otimes \mathbb{N}}$ the product measure on $\Omega := SL(d, \mathbb{Z})^{\mathbb{N}}$. In [6], if the group generated by the support of μ has no abelian subgroup of finite index and acts irreducibly on $\mathbb{R}^d$, a spectral gap is proved for the convolution by μ on L^2_0 and a CLT is deduced for $f(\tau_k(\omega) \ldots \tau_1(\omega)x)$ as a random variable defined on $(\Omega \times \mathbb{T}^d, \mathbb{P} \otimes \lambda)$. Remark that results of Derriennic and Lin [5] imply the CLT for f in L^2_0 not only for the stationary measure of the Markov chain, but also for λ-almost every x, with respect to the measure starting from x. This is a quenched CLT, but with a meaning different from ours: for them x is fixed, for us ω is fixed. Note also the following "quenched" theorem in [6]: for any f in L^2_0, for any $\epsilon > 0$, for $\mathbb{P}$-almost every ω,

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} f(\tau_k(\omega) \ldots \tau_1(\omega) \cdot) = o(\log^{3/2+\epsilon} n).^1$$

Our main result here is the following:

Theorem *Let $\mathcal{A}$ be a proximal and totally irreducible finite set [2] of matrices $d \times d$ with coefficients in $\mathbb{Z}$ and determinant ± 1. Let μ be a probability measure with support $\mathcal{A}$ and $\mathbb{P} = \mu^{\otimes \mathbb{N}}$ be the product measure on $\Omega := \mathcal{A}^{\mathbb{N}}$. Let f be a centered Hölder function on $\mathbb{T}^d$ or a centered characteristic function of a regular set. Then, if $f \not\equiv 0$, for $\mathbb{P}$-almost every ω the limit $\sigma(f) := \lim_n \frac{1}{\sqrt{n}} \|S_n(\omega, f)\|_2$ exists and is positive, and*

$$\left(\frac{1}{\sigma(f)\sqrt{n}} \sum_{k=1}^{n} f(\tau_k(\omega) \ldots \tau_1(\omega) \cdot) \right)_{n \geq 1}$$

converges in distribution to the normal law $\mathcal{N}(0,1)$ with a rate of convergence.

An analogous result has been proved for positive 2×2 matrices and differentiable functions f in [2] via a different method.

The paper is organized as follows. In Section 1 we give sufficient conditions that ensure the approximation by a normal law of the distribution of the normalized sums $\frac{1}{\|S_n\|_2} S_N f$. The proof is based on the method of multiplicative systems (cf. Komlòs [11]) (see B. Petit [14] for an other application of this method). In Section 2, we address the case of a product of independent matrices and prove a "quenched" CLT as mentioned above.

[1]This is true in a more general abstract situation (see [6])

[2] The result is still true if $\mathcal{A}$ is proximal, irreducible on $\mathbb{R}^d$ and the semigroup generated by $\mathcal{A}$ coincide with the group generated by $\mathcal{A}$.

The key inequalities are deduced from results of Guivarc'h and Raugi ([10], [8]). Section 3 is devoted to the general stationary case under stronger assumptions on the set $\mathcal{A}$, in particular for 2×2 positive matrices.

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Contents

1 Preliminaries

1.1 A criterion of Komlòs

Notations Let d be an integer ≥ 2 and $\|\cdot\|$ be the norm on $\mathbb{R}^d$ defined by $\|x\| = \max_{1\leq i\leq d}|x_i|, x \in \mathbb{R}^d$. We denote by $d(x,y) := \inf_{n\in\mathbb{Z}^d}(\|x-y-n\|)$ the distance on the torus. The characters of the torus, $t \to e^{2\pi i\langle n,t\rangle}$ for $n = (n_1,...,n_d) \in \mathbb{Z}^d$, are denoted by $\chi(n,t)$ or by $\chi_n(t)$.

The Fourier coefficients of a function $h \in L^2(\mathbb{T}^d)$ are denoted by $(\hat{h}(n), n \in \mathbb{Z}^d)$. The Hölder norm of order α of an α-Hölder function f on the torus is

$$\|f\|_\alpha = \|f\|_\infty + \sup_{x\neq y}\frac{|f(x)-f(y)|}{d(x,y)^\alpha}.$$

A subset E of the torus is said to be regular if there exist $C > 0$ and $\alpha \in]0, 1]$ such that

$$\lambda(\{t \in \mathbb{T}^d : \ d(t, \partial E) \leq \varepsilon\}) \leq C\varepsilon^\alpha, \forall \varepsilon > 0.$$

The Lebesgue measure on $\mathbb{T}^d$ is denoted by λ. It is invariant under the action of automorphisms of the torus. The action of a product of automorphisms $\tau_j ... \tau_i$, $j \geq i$, corresponds to the action on the characters of the matrices which define the automorphisms *by composition on the right side.* If $A_1, A_2, \dots$ is a sequence of matrices, if $i \leq j$ are two positive integers, we use the notation

$$A_i^j := A_i \dots A_j. \tag{1}$$

Multiplicative systems

In the proof of the central limit theorem we will use a lemma on "multiplicative systems" (cf. Komlòs [11]). The quantitative formulation of the result will yields a rate of convergence in the CLT. The proof of the lemma is given in appendix.

Lemma 1.1. *Let u be an integer ≥ 1, $(\zeta_k)_{0 \leq k \leq u-1}$ be a sequence of length u of real bounded random variables, and a be a real positive number. Let us denote, for $x \in \mathbb{R}$:*

$$\begin{aligned} Z(x) &= \exp(ix \sum_{k=0}^{u-1} \zeta_k(.)), \ Q(x, .) = \prod_{k=0}^{u-1} (1 + ix\zeta_k(.)), \\ Y &= \sum_{k=0}^{u-1} \zeta_k^2, \quad \delta = \max_{0 \leq k \leq u-1} \|\zeta_k\|_\infty. \end{aligned}$$

There is a constant C such that, if $|x| \, \delta \leq 1$, then

$$|\mathbb{E}[Z(x)] - e^{-\frac{1}{2} a x^2}| \leq Cu|x|^3\delta^3 + \frac{1}{2}x^2 \|Q(x)\|_2 \|Y - a\|_2 + |1 - \mathbb{E}[Q(x)]|. \tag{2}$$

If moreover $|x| \|Y - a\|_2^{\frac{1}{2}} \leq 1$, then

$$|\mathbb{E}[Z(x)] - e^{-\frac{1}{2} a x^2}| \leq C\, u\, |x|^3\delta^3 + (3 + 2\, e^{-\frac{1}{2} a x^2} \, \|Q(x)\|_2)|x| \|Y - a\|_2^{\frac{1}{2}} + e^{-\frac{1}{2} ax^2} |1 - \mathbb{E}[Q(x)]|. \tag{3}$$

If $\mathbb{E}[Q(x, .)] \equiv 1$, the previous bound reduces to

$$Cu\, |x|^3\delta^3 + (3 + 2\ e^{-\frac{1}{2} a x^2} \, \|Q(x)\|_2) \, |x| \|Y - a\|_2^{\frac{1}{2}}. \tag{4}$$

1.2 Bounding $|\mathbb{E}[e^{ix\frac{S_n}{\|S_n\|_2}}] - e^{-\frac{1}{2}x^2}|$

The application of the lemma to the action on the torus of matrices in $SL(d, \mathbb{Z})$ requires a property of "separation of the frequencies" which is expressed in the following property.

Property 1.2. *Let $(A_1, \ldots, A_n)$ be a finite set of matrices in $SL(d, \mathbb{Z})$ and let (D, Δ) be a pair of positive reals. We say that the property $\mathcal{S}(D, \Delta)$ is satisfied by $(A_1, \ldots, A_n)$ if the following property is satisfied:*

Let s be an integer ≥ 1. Let $1 \leq \ell_1 \leq \ell_1' \leq \ell_2 \leq \ell_2' \leq \ell_3 \leq ... < \ell_s \leq \ell_s' \leq n$ be any increasing sequence of $2s$ integers, such that $\ell_{j+1} \geq \ \ell_j' + \Delta$ for $j = 1, ..., s-1$. Then for every families $p_1, p_2, ..., p_s$ and $p_1', p_2', ..., p_s' \in \mathbb{Z}^d$ such that $A_1^{\ell_s'} p_s' + A_1^{\ell_s} p_s \neq 0$ and $\|p_j\|, \|p_j'\| \leq D$ for $j = 1, ..., s$, we have:

$$\sum_{j=1}^{s} [A_1^{\ell_j'} p_j' + A_1^{\ell_j} p_j] \neq 0. \tag{5}$$

A particular case of the property is the following. Let s be an integer ≥ 1. Let $\ell_1 < \ell_2 < ... < \ell_s$ be any increasing sequence of s integers such that $\ell_{j+1} \geq \ \ell_j + \Delta$ for $j = 1, ..., s-1$. Then for every family $p_1, p_2, ..., p_s \in \mathbb{Z}^d$ such that $p_s \neq 0$ and $\|p_j\| \leq D$ for $j = 1, ..., s$, we have:

$$\sum_{j=1}^{s} A_1^{\ell_j} p_j \neq 0. \tag{6}$$

This condition implies a multiplicative property as shown by the following lemma:

Lemma 1.3. *Let (D, Δ) be such that the property $\mathcal{S}(D, \Delta)$ holds with respect to the finite sequence of matrices $(A_1, \ldots, A_n)$. Let g be a trigonometric polynomial such that $\hat{g}(p) = 0$ for $\|p\| > D$. If $\ell_1 < \ell_2 < ... < \ell_s$ is an increasing sequence of integers such that $\ell_{j+1} \geq \ \ell_j + \Delta$ for $j = 1, ..., s-1$, then*

$$\int \prod_{j=1}^{s} g(\tau_{\ell_j} ... \tau_1 t) \ dt = 0.$$

Recall that the transformation τ_ℓ is associated to the matrix A_ℓ as said in the introduction.

In what follows, relative sizes of D and Δ will be of importance. The interesting case is when $\mathcal{S}(D, \Delta)$ is satisfied with Δ small compared to D (say Δ of order $\ln D$). We will now focus on the characteristic function

$$\mathbb{E}[e^{ixS_n}] = \int_{\mathbb{T}^d} e^{ix \sum_{\ell=0}^{n-1} g_n(\tau_\ell ... \tau_1 t)} \, dt$$

for a real trigonometric polynomial g_n, where S_n are the ergodic sums

$$S_n(t) = \sum_{\ell=1}^{n} g_n(\tau_\ell ... \tau_1 t).$$

We will use the inequality given by the following lemma for large integers n. Typically, if $\|S_n\|_2$ is of order $\sqrt{n}$, it will be applied when Δ_n is small compared with $n^{\beta/2}$.

Lemma 1.4. *Let n be an integer. Suppose that there exist $\beta \in]0,1[$, $D_n > 0$, $\Delta_n > 0$ such that $\Delta_n < \frac{1}{2}n^\beta$ and $\mathcal{S}(D_n, \Delta_n)$ is satisfied with respect to the finite sequence of matrices $(A_1, \dots, A_n)$. Then, if g_n is a real trigonometric polynomial with $\hat{g}_n(p) = 0$ for $\|p\| > D_n$, there exists a polynomial function C with positive coefficients such that, for $|x|\|g_n\|_\infty n^\beta \leq \|S_n\|_2$ and $|x|\|g_n\|_\infty^{1/2} n^{\frac{1+3\beta}{4}} \leq \|S_n\|_2$:*

$$\begin{aligned} &|\mathbb{E}[e^{ix\frac{S_n}{\|S_n\|_2}}] - e^{-\frac{1}{2}x^2}| \\ \leq\ & C(\|g_n\|_\infty^{1/2})[|x|\|S_n\|_2^{-1}\Delta_n n^{\frac{1-\beta}{2}} + |x|^3\|S_n\|_2^{-3}n^{1+2\beta} + |x|\|S_n\|_2^{-1}n^{\frac{1+3\beta}{4}} \\ &+ |x|^2\|S_n\|_2^{-1}n^{\frac{1-\beta}{2}}\Delta_n + |x|^2\|S_n\|_2^{-2}n^{1-\beta}\Delta_n^2]. \end{aligned} \tag{7}$$

Proof The proof of (7) is given in several steps.

A) *Replacement of S_n by a sum with "gaps"*

In order to apply Lemma 1.1, we replace the sums S_n by a sum of blocks separated by an interval of length Δ_n.

Let $\beta \in]0,1[$, D_n, Δ_n and g_n as in the statement of the lemma. We set:

$$v_n := \lfloor n^\beta \rfloor,\ u_n := \lfloor n/v_n \rfloor \leq 2n^{1-\beta}, \tag{8}$$

$$L_{k,n} := kv_n,\ R_{k,n} := (k+1)v_n - \Delta_n, \tag{9}$$

$$I_{k,n} := [L_{k,n},\ R_{k,n}],\ \text{for } 0 \leq k \leq u_n - 1. \tag{10}$$

Let $S'_n(t)$ be the sum with "gaps":

$$S'_n(t) := \sum_{k=0}^{u_n-1} T_{k,n}(t), \tag{11}$$

where

$$T_{k,n}(t) := \sum_{L_{k,n} < \ell \leq R_{k,n}} g_n(\tau_\ell \dots \tau_1 t). \tag{12}$$

The interval $[1,n]$ is divided into u_n blocks of length $v_n - \Delta_n$. The number of blocks is almost equal to $n^{1-\beta}$ and their length almost equal to n^β. The integers $L_{k,n}$ and $R_{k,n}$ are respectively the left and right ends of the blocks, which are separated by intervals of length Δ_n. The array of r.v.'s $(T_{k,n})$ is a "multiplicative system" in the sense of Komlòs.

Expression of $|T_{k,n}(t)|^2$

$$\begin{aligned} |T_{k,n}(t)|^2 &= (\sum_{\ell' \in I_{k,n}} \sum_{p' \in \mathbb{Z}^d} \hat{g}(p')\chi(A_1^{\ell'}p', t))\,(\sum_{\ell \in I_{k,n}} \sum_{p \in \mathbb{Z}^d} \overline{\hat{g}(p)}\chi(-A_1^\ell p, t)) \\ &= \sum_{p,p' \in \mathbb{Z}^d} \sum_{\ell,\ell' \in I_{k,n}} \hat{g}(p')\overline{\hat{g}(p)}\chi(A_1^{\ell'}p' - A_1^\ell p, t) \\ &= \sigma_{k,n}^2 + W_{k,n}(t), \end{aligned}$$

with

$$\begin{aligned} \sigma_{k,n}^2 &:= \int |T_{k,n}(t)|^2 dt = \sum_{p,p' \in \mathbb{Z}^d} \hat{g}(p')\overline{\hat{g}(p)} \sum_{\ell,\ell' \in I_{k,n}} 1_{A_1^{\ell'} p' = A_1^{\ell} p}, \qquad (13) \\ W_{k,n}(t) &:= \sum_{p,p' \in \mathbb{Z}^d} \hat{g}(p')\overline{\hat{g}(p)} \sum_{\ell,\ell' \in I_{k,n} : A_1^{\ell'} p' \neq A_1^{\ell} p} \chi(A_1^{\ell'} p' - A_1^{\ell} p, t). \end{aligned}$$

B) *Application of Lemma 1.1* We will now apply Lemma 1.1 to the array of r.v.'s $(T_{k,n}, 0 \leq k \leq u_n - 1)$. For a fixed n, we use the same notations as in the lemma: $u = u_n$ and for $k = 0, ..., u_n - 1$

$$\begin{aligned} \zeta_k &= T_{k,n}, \; Y = Y_n = \sum_{k=0}^{u_n - 1} |T_{k,n}|^2, \\ a &= a_n = \mathbb{E}(Y_n) = \sum_k \sigma_{k,n}^2. \end{aligned}$$

With the notation of the lemma, the expression of $Q_n(x,t)$ is

$$Q_n(x,t) = \prod_{k=0}^{u_n - 1} (1 + ixT_{k,n}(t)). \qquad (14)$$

First let us checked that $\mathbb{E}[Q_n(x,.)] = 1, \forall x$. The expansion of the product gives

$$Q_n(x,t) = 1 + \sum_{s=1}^{u_n} (ix)^s \sum_{0 \leq k_1 < ... < k_s \leq u_n - 1} \prod_{j=1}^{s} T_{k_j,n}(t).$$

The products $\prod_{j=1}^s T_{k_j,n}(t)$ are combinations of expressions of the type: $\chi(\sum_{j=1}^s A_1^{\ell_j} p_j, t)$, with $\ell_j \in I_{k_j,n}$ and $\|p_j\| \leq D_n$. So, by the property $\mathcal{S}(D, \Delta)$, $\sum_{j=1}^s A_1^{\ell_j} p_j \neq 0$, and $\int \prod_{j=1}^{s} T_{k_j,n}(t) \, dt = 0$ (cf. Lemma 1.3).

Now we successively bound the quantities involved in Inequality (4).

B1) *Bounding* $u_n \delta_n^3$

$$u_n \delta_n^3 = u_n \max_{0 \leq k \leq u_n - 1} \|T_{k,n}\|_\infty^3 \leq C n^{1-\beta} \|g_n\|_\infty^3 n^{3\beta} = C \|g_n\|_\infty^3 n^{1+2\beta}.$$

B2) *Bounding* $\|Y_n - a_n\|_2$

If $U_1, ..., U_L$ are real square integrable r.v.'s such that

$$\mathbb{E}[(U_k - \mathbb{E}\, U_k)(U_{k'} - \mathbb{E}\, U_{k'})] = 0, \forall 1 \leq k < k' \leq L,$$

then the following inequality holds

$$\begin{aligned}\|\sum_k U_k - \sum_k \mathbb{E}[U_k]\|_2^2 &= \sum_k \mathbb{E}[U_k^2] - (\sum_k \mathbb{E}[U_k])^2 \\ &\leq \sum_k \mathbb{E}[U_k^2] \leq L \max_k \|U_k\|_\infty \max_k \mathbb{E}(|U_k|).\end{aligned}$$

We apply this bound to $U_k = (T_{k,n})^2$ and $L = u_n$ (remark that $T_{k,n}^2 = \sigma_{k,n}^2 + W_{k,n}$ and that we have orthogonality: $\int W_{k,n} W_{k',n} dt = 0$, $1 \leq k < k' < u_n$, due to the choice of the gap. Using rough bounds for $\|T_{k,n}\|_\infty^2$ and $\|T_{k,n}\|_2^2$, it implies

$$\|\sum_k T_{k,n}^2 - \sum_k \sigma_{k,n}^2\|_2^2 \leq u_n \|g_n\|_\infty^2 v_n^2 \max_k \sigma_{k,n}^2 \leq 2\|g_n\|_\infty^2 n^{1+\beta} \max_k \sigma_{k,n}^2.$$

So we have

$$|x|\|Y_n - a_n\|_2^{\frac{1}{2}} \leq 2^{1/4}\|g_n\|_\infty^{1/2} |x| n^{\frac{1+\beta}{4}} \max_k \sigma_{k,n}^{\frac{1}{2}}. \tag{15}$$

B3) *Bounding* $\mathbb{E}|Q_n(x)|^2$

This is the main point. We have

$$\begin{aligned}|Q_n(x,t)|^2 &= \prod_{k=0}^{u_n-1} (1 + x^2 |T_{k,n}(t)|^2) = \prod_{k=0}^{u_n-1} [1 + x^2\sigma_{k,n}^2 + x^2 W_{k,n}(t)] && (16)\\ &= \prod_{k=0}^{u_n-1} [1 + x^2\sigma_{k,n}^2] \prod_{k=0}^{u_n-1} [1 + \frac{x^2}{1 + x^2\sigma_{k,n}^2} W_{k,n}(t)] && (17)\end{aligned}$$

We will show that the integral of the second factor with respect to t is equal to 1. The first factor in (17) is constant and the bound $1 + y \leq e^y, \forall y \geq 0$, implies

$$\prod_{k=0}^{u_n-1} [1 + x^2\sigma_{k,n}^2] \leq e^{x^2 \sum_{k=0}^{u_n-1} \sigma_{k,n}^2} = e^{a_n x^2}.$$

Hence the bound

$$\int |Q_n(x,t)|^2 \, dt \leq e^{a_n x^2}.$$

It remains to show that

$$\int \prod_{k=0}^{u_n-1} [1 + \frac{x^2}{1 + x^2\sigma_{k,n}^2} W_{k,n}(t)] \, dt = 1.$$

In the integral the products $W_{k_1}(t)...W_{k_s}(t)$, $0 \leq k_1 < ... < k_s < u_n$, are linear combinations of expressions of the form

$$\chi(\sum_{j=1}^{s} [A_1^{\ell'_j} p'_j - A_1^{\ell_j} p_j], t),$$

where $\ell_j, \ell'_j \in I_{k_j,n}$, $A_1^{\ell'_j} p'_j \neq A_1^{\ell_j} p_j$, $j = 1, ..., s$ and p_j, p'_j are vectors with integral coordinates and norm $\leq D_n$ which correspond to the non null terms of the trigonometric polynomial g_n.

As $\mathcal{S}(D_n, \Delta_n)$ is satisfied, our choice of gap in the definition of the intervals $I_{k_j,n}$ implies $\sum_{j=1}^s (A_1^{\ell'_j} p'_j - A_1^{\ell_j} p_j) \neq 0$ and so the integral of the second factor in (17) reduces to 1.

From the previous inequalities (in particular (15) and (4) of Lemma 1.1) we deduce that, if $|x| \|g_n\|_\infty n^\beta \leq 1$ and $|x| 2^{1/4} \|g_n\|_\infty^{1/2} n^{\frac{1+\beta}{4}} \max_k \sigma_{k,n}^{\frac{1}{2}} \leq 1$, then

$$\begin{aligned} |\mathbb{E}[e^{ixS'_n}] - e^{-\frac{1}{2}a_n x^2}| &\leq |x|^3 u_n \delta_n^3 + (3 + 2\, e^{-\frac{1}{2}a_n\, x^2}\, \|Q(x)\|_2)\, |x| \|Y - a_n\|_2^{\frac{1}{2}} \\ &\leq C(|x|^3 \|g_n\|_\infty^3 n^{1+2\beta} + |x| \|g_n\|_\infty^{1/2} n^{\frac{1+\beta}{4}} \max_k \sigma_{k,n}^{\frac{1}{2}}). \end{aligned} \tag{18}$$

C) *Bounding the difference between S_n and S'_n*

Recall that S_n is the sum $\sum_1^n g_n(\tau_k ... \tau_1 x)$ and $S'_n = \sum_k T_{k,n}$ is the sum with gaps. We still have to bound the error made when replacing S_n by S'_n:

$$\begin{aligned} \|S_n - S'_n\|_2^2 &= \int |\sum_{k=0}^{u_n - 1} \sum_{R_{k,n} < \ell \leq L_{k+1,n}} g_n(\tau_\ell ... \tau_1 t)|^2 \, dt \\ &= \sum_{k=0}^{u_n-1} \int | \sum_{R_{k,n} < \ell \leq L_{k+1,n}} g_n(\tau_\ell ... \tau_1 t)|^2 \, dt \\ &\quad +2 \sum_{0 < k < k' \leq u_n - 1} \int \sum_{R_{k,n} < \ell \leq L_{k+1,n}} g_n(\tau_\ell ... \tau_1 t) \sum_{R_{k',n} < \ell' \leq L_{k'+1,n}} g_n(\tau_{\ell'} ... \tau_1 t)\, dt. \end{aligned}$$

The length of the intervals for the sums in the integrals is Δ_n. The second sum in the previous expression is 0 by Lemma 1.3 (since $n^\beta - \Delta_n > \Delta_n$). Each integral in the first sum is bounded by $\|g_n\|_\infty^2 \Delta_n^2$. It implies:

$$\|S_n - S'_n\|_2^2 \leq \|g_n\|_2^2\, \Delta_n^2\, u_n \leq 2 \|g_n\|_\infty^2 n^{1-\beta} \Delta_n^2. \tag{19}$$

Thus, we have

$$\begin{aligned} |\|S_n\|_2^2 - \|S'_n\|_2^2| &\leq 2 \|S_n\|_2 \|S_n - S'_n\|_2 + \|S_n - S'_n\|_2^2 \\ &\leq 2\sqrt{2} \|S_n\|_2 \|g_n\|_\infty n^{\frac{1-\beta}{2}} \Delta_n + 2 \|g_n\|_\infty^2 n^{1-\beta} \Delta_n^2. \end{aligned} \tag{20}$$

On an other hand, setting $Z_n(x) = e^{ixS_n}$, $Z'_n(x) = e^{ixS'_n}$, we have:

$$\begin{aligned} |\mathbb{E}[Z_n(x) - Z'_n(x)]| &\leq \mathbb{E}[|1 - e^{ix(S_n - S'_n)}|] \leq |x| \mathbb{E}[|S_n - S'_n|] \leq |x| \|S_n - S'_n\|_2 \\ &\leq \sqrt{2} |x| \|g_n\|_\infty n^{\frac{1-\beta}{2}} \Delta_n. \end{aligned} \tag{21}$$

D) *Conclusion*

Now we gather the previous bounds. Recall that $\sigma_{k,n}^2$ (defined by (13)) is bounded by $C\|g_n\|_\infty^2 n^{2\beta}$.

From (21), (20) and (18), we get that, if $|x|\|g_n\|_\infty n^\beta \leq 1$ and $|x|\|g_n\|_\infty^{1/2} n^{\frac{1+3\beta}{4}} \leq 1$:

$$
\begin{aligned}
& |\mathbb{E}[e^{ixS_n}] - e^{-\frac{1}{2}\|S_n\|_2^2 x^2}| \\
\leq\ & |\mathbb{E}[e^{ixS_n}] - \mathbb{E}[e^{ixS'_n}]| + |\mathbb{E}[e^{ixS'_n}] - e^{-\frac{1}{2}a_n x^2}| + |e^{-\frac{1}{2}a_n x^2} - e^{-\frac{1}{2}\|S_n\|_2^2 x^2}| \\
\leq\ & |\mathbb{E}[e^{ixS_n}] - \mathbb{E}[e^{ixS'_n}]| + C(|x|^3\|g_n\|_\infty^3 n^{1+2\beta} + |x|\|g_n\|_\infty^{1/2} n^{\frac{1+\beta}{4}} \max_k \sigma_{k,n}^{\frac{1}{2}}) + \frac{1}{2}x^2|a_n - \|S_n\|_2^2| \\
\leq\ & C(\|g_n\|_\infty)[|x|\Delta_n n^{\frac{1-\beta}{2}} + |x|^3 n^{1+2\beta} + |x|n^{\frac{1+5\beta}{4}} + |x|^2\|S_n\|_2 n^{\frac{1-\beta}{2}}\Delta_n + |x|^2 n^{1-\beta}\Delta_n^2].
\end{aligned}
$$

Replacing x by $x\|S_n\|_2^{-1}$, we obtain Inequality (7) of the lemma. $\square$

2 Products of independent matrices in $SL(d,\mathbb{Z})$

2.1 Products of matrices (reminders)

Let $\mathcal{A}$ be a finite set of matrices $d \times d$ with coefficients in $\mathbb{Z}$ and determinant ± 1. Let H be the semi-group generated by $\mathcal{A}$.

We assume that there is a contracting sequence in H (proximality). This property holds if $\mathcal{A}$ contains a matrix with a simple dominant eigenvalue. We assume also total irreducibility of H. It means that, for every r, the action of H on the exterior product of $\bigwedge_r \mathbb{R}^d$ has no invariant finite unions of non trivial sub-spaces (cf. [15] for this notion).

Let μ be a probability on $\mathcal{A}$ such that $\mu(\{A\}) > 0$ for every $A \in \mathcal{A}$ and let

$$\Omega := \mathcal{A}^{\mathbb{N}} = \{\omega = (\omega_n), \omega_n \in \mathcal{A}, \forall n \in \mathbb{N}\}$$

be the product space endowed with the product measure $\mathbb{P} = \mu^{\bigotimes \mathbb{N}}$. For every element ω in Ω, we denote by $A_k(\omega)$ (or simply A_k) its k-th coordinate. In other words, we consider a sequence of i.i.d. random variables (A_k) with values in $SL(d,\mathbb{Z})$ and distribution μ, where μ is a discrete probability measure with support $\mathcal{A}$.

In this section, we prove a central limit theorem with a (small) rate of convergence for the action of the product $A_n(\omega)\dots A_1(\omega)$ on the torus for a.a. ω. This establishes, in a more general setting and by a different method, a "quenched" central limit theorem obtained in [2]. The proof relies on results on products of random matrices obtained by Guivarc'h, Le Page and Raugi ([10], [8], [13], [9]). Let us describe the results we will need.

The group $G = SL(d,\mathbb{Z})$ acts on the projective space $\mathbf{P}^{d-1}$. We denote by $(g,x) \to g.x$ the action. For μ a probability measure on G, this define a μ-random walk on $\mathbf{P}^{d-1}$, where the probability for going from x to $g.x$ is $d\mu(g)$ (in our case μ has a finite support $\mathcal{A}$ and $d\mu(g)$ is just $\mu(g)$). Let X and A be two independent random variables, respectively with values in $\mathbf{P}^{d-1}$ and G, and with distribution ν and μ. Then the distribution of $A.X$ is $\mu * \nu$, where

$$(\mu * \nu)(\varphi) = \int_{\mathbf{P}^{d-1}} \int_G \varphi(g.x)\ d\mu(g)\ d\nu(x).$$

The measure ν is μ-stationary if $\mu*\nu = \nu$, i.e., when$X$ and $A.X$ have the same distribution. If $\mathcal{A}$ is proximal and totally irreducible, then there is a unique μ-stationary measure on $\mathbf{P}^{d-1}$ denoted by ν.

Notations Let φ be a function on $\mathbf{P}^{d-1}$. We set

$$[\varphi] = \sup_{u,v \in \mathbf{P}^{d-1}, u \neq v} \frac{|\varphi(u) - \varphi(v)|}{e(u,v)},$$

where e is the distance on $\mathbf{P}^{d-1}$ given by the sinus of the angle between two vectors. A function φ is said to be Lipschitz if $\|\varphi\| = \|\varphi\|_\infty + [\varphi] < \infty$.

For a matrix $M \in Gl(d, \mathbb{R})$, we denote by

$$M = N(M)\ D(M)\ K(M) \tag{22}$$

its Iwasawa decomposition with $N(M)$ an upper triangular matrix and $A(M)$ a diagonal matrix with positive diagonal entries. For $M = A_1(\omega) \dots A_n(\omega)$, we write

$$\begin{aligned} N^{(n)}(\omega) &:= N(A_1(\omega) \dots A_n(\omega)), \\ D^{(n)}(\omega) &:= D(A_1(\omega) \dots A_n(\omega)), \\ K^{(n)}(\omega) &:= K(A_1(\omega) \dots A_n(\omega)), \end{aligned}$$

so that by (22): $A_1^n(\omega) := A_1(\omega) \dots A_n(\omega) = N^{(n)}(\omega)\, D^{(n)}(\omega)\, K^{(n)}(\omega)$.

Let $a_i^{(n)}(\omega) := D_{ii}(A_1^n(\omega))$, $i = 1, ..., d$, be the diagonal coefficients of the diagonal matrix $D^{(n)}$.

Proposition 2.1. *If $\mathcal{A}$ is proximal and totally irreducible, then there exist $\delta > 0$, $C > 0$, and $\rho \in]0,1[$ such that, for every Lipschitz function φ on $\mathbf{P}^{d-1}$,*

$$\|\mathbb{E}(\varphi(A_n \dots A_1.x)) - \nu(\varphi)\|_\infty \leq C\rho^n \|\varphi\|, \tag{23}$$

$$\int_\Omega \left(\frac{a_i^{(n)}(\omega)}{a_{i+1}^{(n)}(\omega)} \right)^\delta d\omega \leq C\rho^n,\ \ \forall i = 1, \cdots, d-1, \tag{24}$$

$$\sup_n \int_\Omega \|N^{(n)}(\omega)\|^\delta\ d\mathbb{P}(\omega) < \infty, \tag{25}$$

$$\sup_{x \in \mathbf{S}^{d-1}} \int_{\mathbf{S}^{d-1}} |\langle x, y \rangle|^{-\delta} d\nu(y) < +\infty. \tag{26}$$

Proof. These statements are consequences of important results of Lepage and Guivarc'h. They can be deduced: (23) from [13], (24), (25), (26) respectively from Theorems 5, 6, 7' of [8]. $\square$

Let us derive some consequences that we will need.

We first remark that almost surely $a_d^{(n)}(\omega) > a_i^{(n)}(\omega)$, for $i \in [1, d-1[$, for n large enough.

The Markov inequality and (24) show that there are constants $C > 0$, $\zeta > 1$, $\xi_0 \in]0,1[$, and a set $\mathcal{E}_n$ of measure $\leq C\xi_0^n$ such that, if ω does not belong to $\mathcal{E}_n$, then

$$a_d^{(n)}(\omega) > \zeta^n a_i^{(n)}(\omega), \ \forall i = 1, \cdots, d-1. \tag{27}$$

Let e_d be the last element of the canonical basis of $\mathbb{R}^d$. As ${}^tN^{(n)}$ is lower triangular we have

$$\begin{aligned}\mathbb{E}(\varphi({}^tA_n \ldots {}^tA_1.e_d)) &= \mathbb{E}(\varphi({}^tK^{(n)}D^{(n)t}N^{(n)}.e_d)) \\ &= \mathbb{E}(\varphi({}^tK^{(n)}.e_d)).\end{aligned}$$

If we consider the set $\{{}^tA, A \in \mathcal{A}\}$ of transposed matrices, the conditions of proximality and irreducibility are also satisfied and we have the same results with a ${}^t\mathcal{A}$-stationary measure ν'. Because of (23) there exists $\beta_0 \in (0,1)$ such that

$$\|\mathbb{E}(\varphi({}^tA_1 \ldots {}^tA_n.e_d)) - \nu'(\varphi)\|_\infty \leq C\beta_0^n\|\varphi\|.$$

But ${}^tA_n \ldots {}^tA_1$ and ${}^tA_1 \ldots {}^tA_n$ have the same distribution. So, for φ a function on $\mathbf{P}^{d-1}$, under the proximality and irreducibility conditions, there exists $\beta_0 \in (0,1)$ such that

$$\|\mathbb{E}(\varphi({}^tK^{(n)}.e_d)) - \nu'(\varphi)\|_\infty \leq C\beta_0^n\|\varphi\|. \tag{28}$$

The probability ν' satisfies the regularity property (26). We deduce that there exist $C > 0$, $\delta > 0$ such that, for $x \in \mathbf{S}^{d-1}$ and $\varepsilon > 0$, we have

$$\nu'\{y \in \mathbf{S}^{d-1} : \ |\langle x, y\rangle| < \varepsilon\} \leq C\varepsilon^\delta. \tag{29}$$

We are now ready to begin our proof.

2.2 Separation of frequencies

Lemma 2.2. *There exist $n_0 \in \mathbb{N}$, $C > 0$, $\alpha \in]0,1[$ and $\beta \in]0,1[$ such that, if $\varepsilon_n \geq \beta^n$, then for $n \geq n_0$ and for every vector x:*

$$\mathbb{P}(\|A_1 \ldots A_n x\| \leq \varepsilon_n\|A_1 \ldots A_n\|\|x\|) \leq C\varepsilon_n^\alpha. \tag{30}$$

Proof Let $(e_1, \ldots, e_d)$ be the canonical basis of $\mathbb{R}^d$. Let x be a unit vector. By expanding x in the orthonormal basis $({}^tK^{(n)}e_1, \ldots, {}^tK^{(n)}e_d)$, we get

$$A_1(\omega) \ldots A_n(\omega)x = \sum_i \langle x, {}^t K^{(n)} e_i \rangle \, a_i^{(n)} N^{(n)} e_i.$$

Hence:

$$\|A_1(\omega) \ldots A_n(\omega)(x)\| \leq \; a_d^{(n)} \; \|N^{(n)}\| \sum_i |\langle x, {}^t K^{(n)} e_i \rangle| \leq \; a_d^{(n)} \; \|N^{(n)}\| \|x\| \leq \; a_d^{(n)} \; \|N^{(n)}\|.$$

In particular the norm $\|A_1(\omega) \ldots A_n(\omega)\|$ is less than $a_d^{(n)} \; \|N^{(n)}\|$.
On the other hand we have

$$\|A_1(\omega) \ldots A_n(\omega)(x)\| \geq a_d^{(n)} |\langle x, {}^t K^{(n)} e_d \rangle| - \|N^{(n)}\| \sum_{i=1}^{d-1} a_i^{(n)}.$$

If the conditions

$$|\langle x, \, {}^tK^{(n)} e_d \rangle| \geq 2\varepsilon_n \|N^{(n)}\|, \; a_d(A_1^n) > \zeta^n a_i(A_1^n), \forall i \in [1, d-1[,$$

are both satisfied, we have

$$\begin{aligned} \|A_1(\omega) \ldots A_n(\omega)(x)\| \; &\geq \; 2\varepsilon_n \|N^{(n)}\| a_d(A_1^n) \; (1 - \frac{\sum_{i=1}^{d-1} a_i^{(n)}}{a_d(A_1^n) \varepsilon_n}) \\ &\geq \; 2\varepsilon_n \|N^{(n)}\| a_d(A_1^n) \; (1 - (d-1)\zeta^{-n} \varepsilon_n^{-1}). \end{aligned}$$

Take $\zeta_0 \in (1, \zeta)$. If $\varepsilon_n \geq \zeta_0^{-n}$ and $(1 - (d-1)\zeta^{-n_0} {\zeta_0}^{n_0}) \geq 1/2$, then for $n \geq n_0$, we have:

$$\|A_1(\omega) \ldots A_n(\omega)(x)\| \geq \varepsilon_n \|N^{(n)}\| a_d(A_1^n) \; \geq \varepsilon_n \, \|A_1(\omega) \ldots A_n(\omega)\|.$$

Thus we have obtained that, for $n \geq n_0$,

$$\begin{aligned} &\mathbb{P}(\|A_1 \ldots A_n x\| \leq \varepsilon_n \|A_1 \ldots A_n\| \|x\|) \leq \mathbb{P}(|\langle x, \, {}^tK^{(n)} e_d \rangle| \\ &\quad \leq 2\varepsilon_n \|N^{(n)}\|) + \mathbb{P}(a_d(A_1^n) \leq \zeta^n a_i(A_1^n)). \end{aligned}$$

This inequality gives the following one for every b_n. We will later chose b_n related to ε_n.

$$\begin{aligned} &\mathbb{P}(\|A_1 \ldots A_n x\| \leq \varepsilon_n \|A_1 \ldots A_n\| \|x\|) \leq \mathbb{P}(|\langle x, \, {}^tK^{(n)} e_d \rangle| \\ &\leq 2\varepsilon_n b_n) + \mathbb{P}(\|N^{(n)}\| \geq b_n) + \mathbb{P}(a_d(A_1^n) \leq \zeta^n a_i(A_1^n)). \end{aligned}$$

Let $x \in \mathbf{P}^{d-1}$ and $\varepsilon > 0$. By convolution one can smooth the indicator function of a "strip" in $\mathbf{P}^{d-1}$. There exists (φ_ε^x) a Lipschitz function on the sphere with values between 0 and

1, such that $\varphi_\varepsilon^x(y) = 1$ on the set $\{y : |\langle x, y\rangle| < 2\varepsilon\}$, $= 0$ on the set $\{y : |\langle x, y\rangle| > 3\varepsilon\}$, and such that $\|\varphi_\varepsilon^x\| < C\varepsilon^{-1}$. Using (28) and (29), we have:

$$\begin{aligned}
& \mathbb{P}(\{\omega : \ |\langle x, {}^t K^{(n)} e_d\rangle| < 2\varepsilon\}) \\
\leq \ & \int \varphi_\varepsilon^x({}^t K(A_1(\omega)\dots A_n(\omega))e_d) d\mathbb{P}(\omega) \leq \int_{\mathbf{S}^{d-1}} \varphi_\varepsilon^x(v) d\nu'(v) + C\beta_0^n \varepsilon^{-1} \\
\leq \ & \nu'\{v : \ |\langle x, v\rangle| < 3\varepsilon\} + C\beta_0^n\varepsilon^{-1} \leq C(3\varepsilon)^\delta + C\beta_0^n \varepsilon^{-1}.
\end{aligned}$$

By taking $\varepsilon = \varepsilon_n b_n$, it follows

$$\mathbb{P}(\{\omega : \ |\langle x, {}^t K^{(n)} e_d\rangle| < 2\varepsilon_n b_n\}) \leq C(3\varepsilon_n b_n)^\delta + C\beta_0^n (\varepsilon_n b_n)^{-1}.$$

On the other hand, we have by (27)

$$\mathbb{P}(\{\omega : \ a_d(A_1^n(\omega)) \leq \zeta^n a_{d-1}(A_1^n(\omega))\}) \leq C\xi_0^n,$$

so that

$$\begin{aligned}
& \mathbb{P}(\|A_1\dots A_n x\| \leq \varepsilon_n \|A_1 \dots A_n\| \|x\|) \\
\leq \ & \mathbb{P}(a_d(A_1^n) \leq \zeta^n a_{d-1}(A_1^n)) + \mathbb{P}(|\langle x, {}^t K^{(n)} e_d\rangle| < 2\varepsilon_n b_n) + \mathbb{P}(\|N^{(n)}\| \geq b_n) \\
\leq \ & C\xi_0^n + C(3\varepsilon_n b_n)^\delta + C\beta_0^n \varepsilon_n^{-1} b_n^{-1} + C b_n^{-\delta}.
\end{aligned}$$

We are looking for $\alpha > 0$ such that the following inequalities hold: $b_n^{-\delta} \leq \varepsilon_n^\alpha$, $(b_n\varepsilon_n)^\delta \leq \varepsilon_n^\alpha$, $\xi_0^n \leq \varepsilon_n^\alpha$ and $\beta_0^n \varepsilon_n^{-1} b_n^{-1} \leq \varepsilon_n^\alpha$. Let us take $\alpha = \delta/3$ and $b_n = \varepsilon_n^{-1/2}$. Then $b_n^{-\delta} \leq \varepsilon_n^\alpha$ and $(b_n\varepsilon_n)^\delta \leq \varepsilon_n^\alpha$. If $\varepsilon_n \geq \beta_0^{\frac{n}{\frac{1}{2}+\alpha}}$ and $\varepsilon_n \geq \xi_0^{n/\alpha}$ then the two other inequalities are satisfied.

So by taking $\beta > \max(\zeta_0^{-1}, \beta_0^{\frac{1}{\frac{1}{2}+\alpha}}, \xi_0^{1/\alpha})$, we obtain (30).

Remark that the bound is uniform with respect to x in $\mathbf{S}^{d-1}$. $\square$

Corollary 2.3. *Let $0 < \gamma_1 < \gamma_2 < 1$. For a.a. ω, there exists $L_1(\omega) < +\infty$ such that, for $n \geq L_1(\omega)$, for every vector p with integral coordinates and norm less than $e^{n^{\gamma_1}}$:*

$$\|A_1^n p\| \geq e^{-n^{\gamma_2}} \|A_1^n\| \|p\|.$$

Proof Let $n \geq n_0$ such that $e^{-n^{\gamma_2}} \geq \beta^n$. By Lemma 2.2, for each vector p we have

$$\mathbb{P}(\|A_1^n p\| \leq e^{-n^{\gamma_2}} \|A_1^n\| \|p\|) \leq C e^{-\alpha n^{\gamma_2}}.$$

Therefore the probability that there is an integral vector p with a norm less than $e^{n^{\gamma_1}}$ such that

$$\|A_1^n p\| \leq e^{-n^{\gamma_2}} \|A_1^n\| \|p\|$$

is less then $C e^{d n^{\gamma_1}} e^{-\alpha n^{\gamma_2}}$. This is the general term of a summable series. We conclude by the Borel-Cantelli lemma. $\square$

Corollary 2.4. *For every $M > 0$, there exists $F > 0$ such that for a.a. ω, there is $L_2(\omega) < +\infty$ such that, for $n \geq L_2(\omega)$, for every vector p with integral coordinates and norm less than n^M:*

$$\|A_1^n p\| \geq \frac{1}{n^F}\|A_1^n\|\|p\|.$$

Proof Let $n \geq n_0$ such that $n^{-F} \geq \beta^n$. Thus by Lemma 2.2, for each vector p, for $n \geq n_0$, we have:

$$\mathbb{P}(\|A_1^n p\| \leq \frac{1}{n^F}\|A_1^n\|\|p\|) \leq C\frac{1}{n^{\alpha F}}.$$

Therefore the probability that there is an integral vector p with a norm less than n^M such that

$$\|A_1^n p\| \leq \frac{1}{n^F}\|A_1^n\|\|p\|$$

is less than $Cn^{dM}\frac{1}{n^{\alpha F}}$. If $\alpha F > dM + 1$ this is the general term of a summable series. We conclude by the Borel-Cantelli lemma. □

Lemma 2.5. *Let $\alpha, \beta \in]0,1[$ given by Lemma 2.2. There exist $n_0 \in \mathbb{N}$, $C > 0$, such that for every unit vector x, every sequence (ε_n) such that $\varepsilon_n \geq \beta^n$, every n and every integer $r = n_0, \cdots, n - n_0$, we have*

$$\mathbb{P}(\|A_1 \dots A_n x\| \leq \varepsilon_r \varepsilon_{n-r}\|A_1 \dots A_r\|\|A_{r+1} \dots A_n\|) \leq C(\varepsilon_r^\alpha + \varepsilon_{n-r}^\alpha).$$

Proof If $\|A_1 \dots A_n x\| > \varepsilon_r\|A_1 \dots A_r\|\|A_{r+1} \dots A_n x\|$ and $\|A_{r+1} \dots A_n x\| > \varepsilon_{n-r}\|A_{r+1} \dots A_n\|$ then $\|A_1 \dots A_n x\| > \varepsilon_r \varepsilon_{n-r}\|A_1 \dots A_r\|\|A_{r+1} \dots A_n\|$. Thus

$$\begin{aligned}&\mathbb{P}(\|A_1 \dots A_n x\| \leq \varepsilon_r \varepsilon_{n-r}\|A_1 \dots A_r\|\|A_{r+1} \dots A_n\|)\\ &\qquad \leq \mathbb{P}(\|A_1^n x\| \leq \varepsilon_r\|A_1^r\|\|A_{r+1}^n x\|) + \mathbb{P}(\|A_{r+1}^n x\| \leq \varepsilon_{n-r}\|A_{r+1}^n\|)\end{aligned}$$

Lemma 2.2 shows that if $\varepsilon_{n-r} \geq \beta^{n-r}$, then, for $n - r \geq n_0$, we have

$$\mathbb{P}(\|A_{r+1}^n x\| \leq \varepsilon_{n-r}\|A_{r+1}^n\|) \leq C\varepsilon_{n-r}^\alpha.$$

The bound obtained in Lemma 2.2 is uniform in x and the matrices A_{r+1}^n and A_1^r are independent. Thus, if $\varepsilon_r \geq \beta^r$ and $r \geq n_0$, one has

$$\begin{aligned}\mathbb{P}(\|A_1^n x\| \leq \varepsilon_r\|A_1^r\|\|A_{r+1}^n x\|) &= \mathbb{P}(\|A_1^r A_{r+1}^n x\| \leq \varepsilon_r\|A_1^r\|\|A_{r+1}^n x\|)\\ &\leq \int_\Omega \mathbb{P}(\|A_1^r y\| \leq \varepsilon_r\|A_1^r\|\|y\|)\, d\mathbb{P}_{A_{r+1}^n x}(y)\\ &\leq C\varepsilon_r^\alpha.\end{aligned}$$

□

Corollary 2.6. *Let $\kappa \in (0,1)$ and $F > \frac{2}{\kappa\alpha}$. For a.e. ω, there is $L_3(\omega) < +\infty$ such that, for $n \geq L_3(\omega)$, for every integer r between n^κ and $n - n^\kappa$:*

$$\|A_1 \dots A_n\| \geq \frac{1}{r^F(n-r)^F}\|A_1 \dots A_r\|\|A_{r+1} \dots A_n\|. \tag{31}$$

<u>Proof</u> If n is large enough and $n^\kappa \leq r \leq n - n^\kappa$, the inequalities $r \geq n_0$ and $r^{-F} \geq \beta^r$ are satisfied. Let x be a unit vector. By Lemma 2.5, for every integer r between n^κ and $n - n^\kappa$, we have

$$\mathbb{P}(\|A_1 \dots A_n x\| \leq r^{-F}(n-r)^{-F}\|A_1 \dots A_r\| \, \|A_{r+1} \dots A_n\|) \leq C(r^{-\alpha F} + (n-r)^{-\alpha F}).$$

Therefore the probability that there is an integer r between n^κ and $n - n^\kappa$ such that

$$\|A_1 \dots A_n x\| \leq r^{-F}(n-r)^{-F}\|A_1 \dots A_r\|\|A_{r+1} \dots A_n\|$$

is less than $2Cn(n^{-\kappa\alpha F})$. By the choice of F we have $\sum_{n=1}^{\infty} n n^{-\kappa\alpha F} < \infty$.
From the Borel-Cantelli lemma, we deduce that, for a.a. ω, there is $L_3(\omega) < +\infty$ such that, for every $n \geq L_3(\omega)$, every integer r between n^κ and $n - n^\kappa$:

$$\|A_1 \dots A_n x\| \geq \frac{1}{r^F(n-r)^F}\|A_1 \dots A_r\|\|A_{r+1} \dots A_n\|.$$

In particular this imples (31). $\square$

Lemma 2.7. *There exists $C_1 > 0$ such that, for a.a. ω, there is $L_4(\omega) < +\infty$ such that, for $n \geq L_4(\omega)$, for every integer $\ell \in [1, n]$, for every $r \in [C_1 \log n, n]$,*

$$\|A_\ell^{\ell+r}\| \geq \zeta^{r(d-1)/d}.$$

<u>Proof</u> We have $\|A_1^r\| \geq |a_d^{(r)}|$. By a previous result, (cf. (27), there exist $\zeta > 1$ and $\xi_0 \in]0,1[$ such that

$$\mathbb{P}(\{|a_d^{(r)}| < \zeta^r |a_i^{(r)}|\}) \leq C\xi_0^r, \forall i = 1, \cdots, d-1.$$

On the other hand the product of the $d_i^{(r)}$'s is one. We thus have:

$$\mathbb{P}(\|A_1^r\| < \zeta^{((d-1)/d)r}) \leq C\xi_0^r. \tag{32}$$

As the probability measure $\mathbb{P}$ is invariant by the shift, for every integer ℓ we have

$$\mathbb{P}(\|A_\ell^{\ell+r}\| < \zeta^{((d-1)/d)r}) \leq C\xi_0^r. \tag{33}$$

The probability that there exist $1 \leq \ell \leq n$, $C_1 \log n \leq r \leq n$ such that $\|A_\ell^{\ell+r}\| < \zeta^{((d-1)/d)r}$ is bounded by $Cn^2\xi_0^{C_1 \log n}$. If $C_1 > -3/\log \xi_0$ the sequence $Cn^2\xi_0^{C_1 \log n}$ is summable. We conclude by the Borel-Cantelli lemma. $\square$

The next proposition on separation of frequencies shows that, for $M > 1$ and $\gamma \in]0,1[$, for a.e. ω, for n big enough, the property $\mathcal{S}(D_n, \Delta_n)$ is satisfied with respect to the finite sequence of matrices $(A_1(\omega), \dots, A_n(\omega))$ for $D_n = n^M$, $\Delta_n = n^\gamma$. It will enable us to use Inequality (7) of Lemma 1.4 with a well chosen γ.

Proposition 2.8. *For every $\gamma \in]0,1[$, every $M \geq 1$, for a.e. ω, there exists a rank $L_5(\omega)$ such that, for every $n \geq L_5(\omega)$, the property $\mathcal{S}(n^M, n^\gamma)$ is satisfied with respect the finite sequence of matrices $(A_1(\omega), \dots, A_n(\omega))$. That is: Let s be an integer ≥ 1. Let $1 \leq \ell_1 \leq \ell'_1 \leq \ell_2 \leq \ell'_2 \leq \dots \leq \ell_s \leq \ell'_s \leq n$ be an increasing sequence of $2s$ integers such that $\ell_{j+1} \geq \ \ell'_j + n^\gamma$, for $j = 1, ..., s-1$. Then, for every $p_1, p_2, ..., p_s$ and $p'_1, p'_2, ..., p'_s \in \mathbb{Z}^d$ such that $A_1^{\ell'_s} p'_s + A_1^{\ell_s} p_s \neq 0$ and $\|p_j\|, \|p'_j\| \leq n^M$ for $j = 1, ..., s$, we have:*

$$\sum_{j=1}^{s} [A_1^{\ell'_j} p'_j + A_1^{\ell_j} p_j] \neq 0. \tag{34}$$

Proof We will use Corollary 2.6 and the gap between the ℓ_j's to obtain a contradiction from the equality

$$A_1^{\ell'_s} p'_s + A_1^{\ell_s} p_s = -\sum_{j=1}^{s-1} [A_1^{\ell'_j} p'_j + A_1^{\ell_j} p_j]. \tag{35}$$

Let us consider two cases:

1) Assume that $\ell'_s - \ell_s$ is small: $0 \leq \ell'_s - \ell_s \leq n^\eta$ (for some $\eta \in (0, \gamma^2)$) or that $p'_s = 0$.

Write $q_s = A_{\ell_s+1}^{\ell'_s} p'_s + p_s$. It is a non zero element of $\mathbb{Z}^d$ by assumption and its norm is bounded by $n^M \max\{\|A\| \ : \ A \in \mathcal{A}\}^{n^\eta} \times n^M \leq 2n^M R^{n^\eta}$. We have $\ell_s > n^\gamma$. So the norm of q_s is bounded by $2\ell_s^{M/\gamma} R^{\ell_s^{\eta/\gamma}}$. According to Corollary 2.3, for every $\eta' > \eta/\gamma$, almost surely, for n large enough, we have

$$\|A_1^{\ell_s} q_s\| \geq e^{-\ell_s^{\eta'}} \|A_1^{\ell_s}\| \geq e^{-n^{\eta'}} \|A_1^{\ell_s}\|. \tag{36}$$

According to Corollary 2.6, we have:

$$\begin{aligned}
\|\sum_{j=1}^{s-1} [A_1^{\ell'_j} p'_j + A_1^{\ell_j} p_j]\| &\leq n^M \sum_{j=1}^{s-1} [\|A_1^{\ell'_j}\| + \|A_1^{\ell_j}\|] \\
&\leq n^M \left[\sum_{j=1}^{s-1} \frac{\|A_1^{\ell_s}\|}{\|A_{\ell'_{j+1}}^{\ell_s}\|} \ell_j'^F (\ell_s - \ell'_j)^F + \sum_{j=1}^{s-1} \frac{\|A_1^{\ell_s}\|}{\|A_{\ell_{j+1}}^{\ell_s}\|} \ell_j^F (\ell_s - l_j)^F \right] \\
&\leq n^{M+2F} \|A_1^{\ell_s}\| \left[\sum_{j=1}^{s-1} \frac{1}{\|A_{\ell'_{j+1}}^{\ell_s}\|} + \sum_{j=1}^{s-1} \frac{1}{\|A_{\ell_{j+1}}^{\ell_s}\|} \right] \\
&\leq n^{M+2F} \|A_1^{\ell_s}\| \left[\sum_{j=1}^{s-1} \zeta^{(d-1)(\ell'_j - \ell_s)/d} + \sum_{j=1}^{s-1} \zeta^{(d-1)(\ell_j - \ell_s)/d} \right].
\end{aligned}$$

The last inequality holds because of Lemma 2.7. As $\ell_j - \ell_s \geq (j-s)n^\gamma$ and $\ell'_j - \ell_s \geq (j-s)n^\gamma$, it implies:

$$\|\sum_{j=1}^{s-1} [A_1^{\ell'_j} p'_j + A_1^{\ell_j} p_j]\| \leq C n^{M+2F} \zeta^{-(d-1)n^\gamma/d} \|A_1^{\ell_s}\|. \tag{37}$$

If we take γ, η and η' such that $\eta/\gamma < \eta' < \gamma$, the inequalities (36) and (37) show that (35) is not satisfied for large n.

2) Now assume that $\ell'_s - \ell_s$ is large: $\ell'_s - \ell_s \geq n^\eta$ and $p'_s \neq 0$.
On the one hand, we have (Corollary 2.4)

$$\|A_1^{\ell'_s} p'_s\| \geq \frac{1}{n^F} \|A_1^{\ell'_s}\|.$$

On the other hand, by Corollary 2.6 and Lemma 2.7:

$$\|A_1^{\ell_s} p_s\| \leq \|p_s\| \frac{\|A_1^{\ell'_s}\|}{\|A_{\ell_{s+1}}^{\ell'_s}\|} \ell_s^F (\ell'_s - \ell_s)^F \leq n^{M+2F} \zeta^{-(d-1)n^\eta/d} \|A_1^{\ell'_s}\|, \tag{38}$$

thus by (37) and (38):

$$\|A_1^{\ell_s} p_s + \sum_{j=1}^{s-1} [A_1^{\ell'_j} p'_j + A_1^{\ell_j} p_j]\| \leq C n^{M+2F} (\zeta^{-(d-1)n^\gamma/d} + \zeta^{-(d-1)n^\eta/d}) \|A_1^{\ell'_s}\|.$$

In this case as above, Equality (35) does not hold for n large enough. $\square$

The proposition implies that, for every increasing sequence of s integers, $\ell_1 < \ell_2 < ... < \ell_s < n$ with $\ell_{j+1} \geq \ \ell_j + n^\gamma$, for $j = 1, ..., s-1$, every $p_1, p_2, ..., p_s \in \mathbb{Z}^d$ such that $p_s \neq 0$ and $\|p_j\| \leq n^M$ for $j = 1, ..., s$, we have:

$$\sum_{j=1}^{s} A_1^{\ell_j} p_j \neq 0; \tag{39}$$

Lemma 2.9. *There exist $\zeta_1 > 1$, $\xi_1 \in]0, 1[$ and $C > 0$ such that*

$$\mathbb{P}(\{\forall p \in \mathbb{Z}^d, \|p\| \leq \zeta_1^n \ : \ \|A_1^n p\| > \zeta_1^n\}) \geq 1 - C\xi_1^n. \tag{40}$$

Proof We have $\|A_1^n\| \geq |a_d^{(n)}|$. As we have seen (Lemma 2.7), there exist $\zeta > 1$ and $\xi_0 \in]0, 1[$ such that

$$\mathbb{P}(\|A_1^n\| \geq \zeta^{((d-1)/d)n}) \geq 1 - C\xi_0^n. \tag{41}$$

According to Lemma 2.2, if ξ_2 is in $]\beta, 1[$, there exist $C > 0$ and $\xi_3 \in]0, 1[$ such that:

$$\mathbb{P}(\|A_1^n p\| \geq \xi_2^n \|A_1^n\| \|p\|) \geq 1 - C\xi_3^n. \tag{42}$$

From (41) and (42) we deduce that, if p is an integral vector, there exist $C > 0$, $\zeta_2 > 1$ and $\xi_4 \in]0, 1[$ such that (take $\xi_2^{-1} < \zeta^{((d-1)/d)}$):

$$\mathbb{P}(\|A_1^n p\| \geq \zeta_2^n) \geq \mathbb{P}(\|A_1^n p\| \geq \zeta_2^n \|p\|) \geq 1 - C\xi_4^n, \tag{43}$$

or equivalently $\mathbb{P}(\|A_1^n p\| \leq \zeta_2^n) \leq C\xi_4^n$.

Let ζ_1 be a real number in $]1, \zeta_2[$ such that $\zeta_1^d \xi_4 < 1$. By taking the sum in the previous inequality over integral vectors p in the ball centered at zero of radius ζ_1^n, we obtain

$$\mathbb{P}(\{\exists p \in \mathbb{Z}^d, \|p\| \leq \zeta_1^n \ : \ \|A_1^n p\| \leq \zeta_2^n\}) \leq C \zeta_1^{dn} \xi_4^n.$$

The lemma follows since $\zeta_2 > \zeta_1$. □

The following lemma will be used in the approximation of a function by a trigonometric polynomial. It can be proved by taking the sequence (φ_n) of the products of the Fejèr kernels in each coordinate.

Proposition 2.10. *There exist a constant $C > 0$ and a sequence of trigonometric polynomials (φ_n) of order less than $d\,n$, such that, for every α-Hölder function f on the torus,*

$$\|\varphi_n * f - f\|_\infty < C \|f\|_\alpha n^{-\alpha},$$

and for every α-regular subset A of the torus, $\|\varphi_n * 1_A - 1_A\|_2 < C n^{-\alpha}$.

2.3 Variance and CLT

We denote by θ the left shift on Ω:

$$\theta : (A_k(\omega))_{k \geq 1} \longmapsto (A_{k+1}(\omega))_{k \geq 1},$$

$\tau_{A_1}(\omega)$ the map on the torus

$$\tau_{A_1(\omega)} : x \longmapsto A_1(\omega) x,$$

θ_τ the transformation on $\Omega \times \mathbb{T}^d$:

$$\theta_\tau : ((A_k(\omega))_{k \geq 1}, x) \longmapsto ((A_{k+1}(\omega))_{k \geq 1}, \tau_{A_1(\omega)} x),$$

and let

$$S_n(\omega, f)(x) := \sum_{k=1}^{n} f(\tau_{A_k(\omega)} \dots \tau_{A_1(\omega)} x).$$

Proposition 2.11. *Let f be a Hölder function on the torus not a.e. null and with zero mean. Then for $\mathbb{P}$-almost every $\omega \in \Omega$ the sequence $(n^{-\frac{1}{2}} \|S_n(\omega, f)\|_2)$ has a limit $\sigma(f)$ which is positive and does not depend on ω.*

Proof Denoting $F(\omega, t) := f(t)$, we have $S_n(\omega, f)(t) = \sum_{k=0}^{n-1} F(\theta_\tau^k(\omega, t))$ and

$$\frac{1}{n} \|S_n(\omega, f)\|^2 = \|f\|^2 + \frac{2}{n} \sum_{r=1}^{n-1} \sum_{\ell=0}^{n-1-r} \int_{\mathbb{T}^d} F(\theta_\tau^\ell(\omega, t)) F(\theta_\tau^{\ell+r}(\omega, t))\, dt. \tag{44}$$

Under our hypotheses, the "global variance" exists: the following convergence holds

$$\lim_n \frac{1}{n} \int \int |S_n(\omega, f)|^2 \, dt \, d\omega = \sigma(f)^2.$$

Actually this holds for every centered function f in $L_0^2(\mathbb{T}^d)$, and we have

$$\sigma(f)^2 = \sum_{r=1}^{\infty} \int_{\Omega\times\mathbb{T}^d} (F\ F\circ\theta_\tau^r)(\omega,t)\,dt\,d\omega \tag{45}$$

as a consequence of the existence of a spectral gap for the operator of convolution by μ on $L_0^2(\mathbb{T}^d)$, which implies the convergence of the series. Moreover it can be shown that $\sigma(f) > 0$ if f is not a.e. null (cf. [7], see also [6]). We have to prove that (44) has the same limit $\sigma(f)^2$ for a.e. ω.

Let us first consider the sum

$$\frac{2}{n}\sum_{r=1}^{n^\alpha}\sum_{\ell=0}^{n-1-r}\int_{\mathbb{T}^d} F(\theta_\tau^\ell(\omega,t))F(\theta_\tau^{\ell+r}(\omega,t))\,dt.$$

The second term of the right-hand part of the following equality

$$\frac{2}{n}\sum_{r=1}^{n^\alpha}\sum_{\ell=0}^{n-1-r}\int_{\mathbb{T}^d} F(\theta_\tau^\ell(\omega,t))F(\theta_\tau^{\ell+r}(\omega,t))\,dt =$$
$$2\sum_{r=1}^{n^\alpha}\int_{\mathbb{T}^d}\frac{1}{n}\sum_{\ell=0}^{n-1}(F\ F\circ\theta_\tau^r)(\theta_\tau^\ell(\omega,t))\,dt - 2\sum_{r=1}^{n^\alpha}\frac{1}{n}\sum_{\ell=n-r}^{n-1}\int_{\mathbb{T}^d} F(\theta_\tau^l(\omega,t))F(\theta_\tau^{\ell+r}(\omega,t))\,dt$$

is bounded by $2\|f\|_2^2 n^{2\alpha-1}$. For $\alpha < 1/2$, it suffices to consider the first term.

Let us denote by ψ_j the function defined on Ω by

$$\psi_j = \int_{\mathbb{T}^d} F\ F\circ\theta_\tau^j\ dt - \int_{\Omega\times\mathbb{T}^d} F\ F\circ\theta_\tau^j\ dt\,d\omega. \tag{46}$$

It only depends on the j first coordinates of ω, so that

$$\int_\Omega \psi_j\ \psi_j\circ\theta^l\ \ d\mathbb{P}(\omega) = 0,\ \text{ if } l > j.$$

We claim that, for every $0 < \alpha < 1$, $j < n^\alpha$, $\eta \geq 2\alpha$,

$$\mathbb{E}[(\sum_{m=0}^{n-1}\psi_j\circ\theta^m)^4] < Cj^2n^2 < Cn^{2+\eta}.$$

To prove it, let us expand the fourth power of $\sum_{m=0}^{n-1}\psi_j\circ\theta^m$:

$$(\sum_{m=0}^{n-1}\psi_j\circ\theta^m)^4 = \sum_{i,k,l,m}\psi_j\circ\theta^i\,\psi_j\circ\theta^k\,\psi_j\circ\theta^l\,\psi_j\circ\theta^m.$$

The number of 4-uples (i,k,l,m) such that $i<k\leq i+j$ and $l<m\leq l+j$ is less than j^2n^2 and, if $i+j<k$ or $l+j<m$, then the integral of the corresponding term is equal to zero. For every $\varepsilon>0$, the probability

$$\mathbb{P}(\sup_{j=1,\dots,n^\alpha}|\sum_{m=0}^{n-1}\psi_j\circ\theta^m|>n^\beta\varepsilon)$$

is less than $\sum_{j=1}^{n^\alpha}\mathbb{E}((\sum_{m=0}^{n-1}\psi_j\circ\theta^m)^4)/\varepsilon^4n^{4\beta}$, therefore it is less than $Cn^{2+\eta+\alpha-4\beta}$. We can chose α, β, η such that this sequence is summable and $\alpha<1-\beta$. Let us take $\beta=0.8$, $\alpha=0.01$ and $\eta=0.02$).

Then almost surely, we have:

$$\lim_n\frac{1}{n^\beta}\sup_{j=1,\dots,n^\alpha}|\sum_{l=0}^{n-1}\psi_j\circ\theta^l|=0. \tag{47}$$

We deduce from (46) that:

$$\begin{aligned}
&\sum_{r=1}^{n^\alpha}\int_{\mathbb{T}^d}\frac{1}{n}\sum_{\ell=0}^{n-1}(F\ F\circ\theta_\tau^r)(\theta_\tau^\ell(\omega,t))\,dt\\
=\ &\sum_{r=1}^{n^\alpha}\frac{1}{n}\sum_{\ell=0}^{n-1}\psi_r\circ\theta^l+\sum_{r=1}^{n^\alpha}\int_{\Omega\times\mathbb{T}^d}\frac{1}{n}\sum_{\ell=0}^{n-1}(F\ F\circ\theta_\tau^r)(\theta_\tau^\ell(\omega,t))\,dt\,d\omega\\
=\ &\sum_{r=1}^{n^\alpha}n^{\beta-1}\frac{1}{n^\beta}\sum_{\ell=0}^{n-1}\psi_r\circ\theta^l+\sum_{r=1}^{n^\alpha}\int_{\Omega\times\mathbb{T}^d}(F\ F\circ\theta_\tau^r)(\omega,t)\,dt\,d\omega.
\end{aligned}$$

By (47) both sequences

$$\begin{aligned}
&\sum_{r=1}^{n^\alpha}\int_{\mathbb{T}^d}\frac{1}{n}\sum_{\ell=0}^{n-1-r}(F\ F\circ\theta_\tau^r)(\theta_\tau^\ell(\omega,t))\,dt,\\
&\sum_{r=1}^{n^\alpha}\int_{\Omega\times\mathbb{T}^d}(F\ F\circ\theta_\tau^r)(\omega,t)\,dt\,d\omega
\end{aligned}$$

converge toward the sum of (45).

We now consider the sum

$$\sum_{r=n^\alpha+1}^{n-1}\int_{\mathbb{T}^d}\frac{1}{n}\sum_{\ell=0}^{n-1-r}(F\ F\circ\theta_\tau^r)(\theta_\tau^\ell(\omega,t))\,dt.$$

For a given integer n, let g_n be a centered polynomial of degree less than n^M such that $\|f-g_n\|_2<n^{-4}$.

Corollary 2.6 shows that, almost surely, the frequencies of the polynomial $g_n \circ \theta_\tau^{l+r}$ are greater than $r^{-F} l^{-F} \|A_1^l\| \|A_{l+1}^{l+r}\|$, so greater than $n^{-2F}\|A_1^l\|\zeta^r$. The norms of the frequencies of the polynomial $g_n \circ \theta_\tau^l$ are less than $n^M \|A_1^l\|$. So, if $r > n^\alpha$, almost surely for a sufficiently large n, $\int_{\mathbb{T}^d} \frac{1}{n} \sum_{\ell=0}^{n-1} (g_n \; g_n \circ \theta_\tau^r)(\theta_\tau^\ell(\omega, t))\, dt = 0$.

Thus, we have almost surely in ω

$$\begin{aligned} \lim_n \frac{\|S_n(\omega, f)\|^2}{n} &= \|f\|^2 + 2\sum_{r=1}^{\infty} \int_{\Omega\times\mathbb{T}^d} (F \; F\circ\theta_\tau^r)(\omega, t)\, dt\, d\omega \\ &= \lim_n \int_{\Omega\times\mathbb{T}^d} \frac{1}{n} [\sum_{k=0}^{n-1} F(\theta_\tau^k(\omega, t))]^2 \; dtd\omega. \end{aligned}$$

□

Remark that the statement of the previous proposition holds if f is a centered characteristic function of a regular set of positive measure of the torus.

The proof the CLT for $S_n(\omega, f)$ is now an application of the method of Section 1.

Theorem 2.12. *Let $\mathcal{A}$ be a proximal and totally irreducible finite set of matrices $d \times d$ with coefficients in $\mathbb{Z}$ and determinant ± 1. Let f be a centered Hölder function on $\mathbb{T}^d$ or a centered characteristic function of a regular set. Then, if $f \not\equiv 0$, for almost every ω the limit $\sigma(f) = \lim_n \frac{1}{\sqrt{n}}\|S_n(\omega, f)\|_2$ exists and is positive, and*

$$(\frac{1}{\sigma(f)\sqrt{n}} \sum_{k=1}^{n} f(\tau_k(\omega)\ldots\tau_1(\omega)\cdot))_{n\geq 1}$$

converges in distribution to the normal law $\mathcal{N}(0,1)$ with a rate of convergence.

Proof Recall that the transformation τ_k is the action on the torus defined by the transposed matrice of A_k.

Let f be a centered Hölder function or a centered characteristic function of a regular set such that $\sigma(f) \neq 0$. We have shown above that almost surely $\|S_n f\|_2$ is equivalent to $\sigma(f)\sqrt{n}$. It suffices to prove convergence of $\frac{S_n f}{\|S_n f\|_2}$ towards the normal law $\mathcal{N}(0,1)$.

There exists an integer M such that, for every n, there is a trigonometric polynomial g_n of degree less than n^M, such that

$$\|S_n f - S_n g_n\|_2 \leq n^{-4}.$$

Therefore $|\mathbb{E}[e^{ix\frac{S_n f}{\|S_n f\|_2}}] - \mathbb{E}[e^{ix\frac{S_n g_n}{\|S_n g_n\|_2}}]|$ tends to 0 and $\frac{\|S_n g_n\|_2}{\sqrt{n}}$ tends to $\sigma(f) \neq 0$. Almost surely for n big enough, the norm $\|S_n g_n\|_2$ is greater than $\frac{1}{2}\sigma(f) n^{1/2}$.

Now we use the notations introduced in Subsection 1.1. According to Proposition 2.8, we can apply Inequality (7) of Lemma 1.4 with $\Delta_n = n^\gamma$ (this is possible if $\gamma < \beta$): for $|x|\|g_n\|_\infty n^\beta \leq \|S_n\|_2$ and $|x|\|g_n\|_\infty^{1/2} n^{\frac{1+3\beta}{4}} \leq \|S_n\|_2$

$$\begin{aligned} & |\mathbb{E}[e^{ix\frac{S_n g_n}{\|S_n g_n\|_2}}] - e^{-\frac{1}{2}x^2}| \\ \leq \; & C(\|g_n\|_\infty)[|x| n^{-\frac{\beta}{2}} + |x|^3 n^{-(-2\beta+1/2)} + |x| n^{-\frac{(-3\beta+1)}{4}} + |x|^2 n^{-\frac{\beta}{2}} n^\gamma + |x|^2 n^{-\beta} n^{2\gamma}] \end{aligned} \quad (48)$$

Here the sequence ($\|g_n\|_\infty$) is bounded. So, by taking β and γ such that $0 < 2\gamma < \beta < 1/4$, we obtain that $|\mathbb{E}[e^{ix\frac{S_n g_n}{\|S_n g_n\|_2}}] - e^{-\frac{1}{2}x^2}|$ tends to 0 for every x.

Using Esseen's inequality it can be shown that there is at least a rate of convergence of order $n^{-1/40}$ in the CLT (cf. 3.14 in the next section). $\square$

Remarks 2.13. 1) If $\mathcal{A}$ reduces to a single ergodic matrix A, the system is clearly not totally irreducible. Nevertheless, as it is well known, the CLT holds in this case.

2) Let us take two matrices A and A^{-1} with a non uniform probability, then the system is not totally irreducible, but we can show that the quenched CLT holds.

4) If we take two matrices A and A^{-1} with equal probability 1/2, then the CLT does not hold for the global system. We get a sort of T, T^{-1} transformation and another limit theorem (see [12]). This makes us think that CLT for a.a. sequence of matrices could not be true.

3 Stationary products, matrices in $SL(2, \mathbb{Z}^+)$

In this section we consider the case of a sequence (A_k) generated by a stationary process. This is more general than the independent stationary case, but we have to assume the rather strong Condition 3.2 below, a condition which is satisfied by matrices in $SL(2, \mathbb{Z}^+)$. In this case some information about the non-nullity of the variance can also be obtained. We will express the stationarity by using the formalism of skew products.

3.1 Ergodicity, decorrelation

We consider an ergodic dynamical system (Ω, μ, θ), where θ is an invertible measure preserving transformation on a probability space (Ω, μ). We denote by X the torus $\mathbb{T}^d$ and by τ_A the automorphism of X associated to a matrix $A \in SL(d, \mathbb{Z})$. Let $\mathcal{A}$ be a finite set of matrices in $SL(d, \mathbb{Z})$.

Notations 3.1. Let $\omega \to A(\omega)$ be a measurable map from Ω to $\mathcal{A}$, and τ the map $\omega \to \tau(\omega) = \tau_{A(\omega)}$. The skew product θ_τ is defined on the product space $\Omega \times X$ equipped with the product measure $\nu := \mu \times \lambda$ by

$$\theta_\tau : \Omega \times X \to \Omega \times X;\ (\omega, t) \mapsto (\theta\omega, \tau(\omega)t).$$

Let F be a function in $L^2(\Omega \times X)$ and, for $p \in \mathbb{Z}^d$, let $F_p(\omega)$ be its Fourier coefficient of order p with respect to the variable t. F can be written:

$$F(\omega, t) = \sum_{p \in \mathbb{Z}^d} F_p(\omega)\chi(p, t),$$

with $\sum_{p\in\mathbb{Z}^d}\int |F_p(\omega)|^2 \ d\mu(\omega) < \infty$.

Let $\mathcal{H}_\alpha^0$ be the set of α-Hölder functions on the torus with null integral. This notation is extended to functions $f(\omega,t)$ on $\Omega \times X$ which are α-Hölder in the variable t, uniformly with respect to ω.

For $k \geq 1$, $j \geq i$, $\omega \in \Omega$, $f \in \mathrm{L}^2(X,\mathbb{R})$, we write

$$\begin{aligned}
\tau(k,\omega) &= \tau(\theta^{k-1}\omega)\dots\tau(\omega),\\
A_i^j(\omega) &= A(\theta^i\omega)A(\theta^{i+1}\omega)...A(\theta^j\omega),\\
S_n(\omega,f)(t) &= \sum_{k=1}^n f(\tau(k,\omega)t).
\end{aligned}$$

In what follows in this subsection and in subsection 3.2, we assume the following condition 3.2 which implies an exponential decay of correlation:

Condition 3.2. *There are constants $C>0$, $\delta>0$ and $\lambda>1$ such that*

$$\forall r\geq 1, \forall A_1,...,A_r \in \mathcal{A},\ \forall p \in \mathbb{Z}^d\setminus\{0\}, \|A_1...A_r p\| \geq C\|p\|^{-\delta}\lambda^r.$$

Proposition 3.3. *Under Condition 3.2, the system $(\Omega\times X, \theta_\tau, \mu\otimes\lambda)$ is mixing on the orthogonal of the subspace of functions depending only on ω. For the skew product map the mixing property holds with an exponential rate on the space of Hölderian functions. If (Ω,μ,θ) is ergodic, then the dynamical system $(\Omega\times X, \theta_\tau, \mu\otimes\lambda)$ is ergodic.*

Proof Let G be in $L^2(\Omega\times X)$ a trigonometric polynomial with respect to t for every ω and such that $G(\omega,t) = \sum_{0<\|p\|\leq D} G_p(\omega)\chi(p,t)$, for a real $D\geq 1$. We have:

$$\begin{aligned}
\langle G\circ\theta_\tau^n, G\rangle_\nu &= \int\int (\sum_p G_p(\theta^n\omega)\chi(A_1^n(\omega)p,t))(\overline{\sum_q G_q(\omega)\chi(q,t)})\ dt\ d\mu(\omega)\\
&= \sum_{p,q}\int\int\ G_p(\theta^n\omega)\chi(A_1^n(\omega)p,t))(\overline{G_q(\omega)\chi(q,t)})\ dt\ d\mu(\omega)\\
&= \sum_{p,q}\int\ G_p(\theta^n\omega)\,\overline{G_q(\omega)}\ 1_{A_1^n(\omega)p=q}\ d\mu(\omega).
\end{aligned}$$

According to Condition 3.2, there is a constant C_1 not depending on D such that $A_1^n(\omega)p\neq q$, for $n\geq C_1\ln D$. Thus we have $\langle G\circ\theta_\tau^n, G\rangle = 0$, for $n \geq C_1 \ln D$.

With a density argument this shows that $\lim_n\langle G\circ\theta_\tau^n, G\rangle_\nu = 0$ for a function G which is orthogonal in $L^2(\nu)$ to functions depending only on ω (with an exponential rate of decorrelation for Hölderian functions in this subspace). If the system (Ω,μ,θ) is ergodic, this implies ergodicity of the extension. $\square$

We are going to prove that, for a.e. ω, the sequence $(n^{-\frac12}\|S_n(\omega,f)\|_2)$, converges to a limit. The norm $\|S_n(\omega,f)\|_2$ is taken with respect to the variable t, ω being fixed.

Proposition 3.4. *For every $f \in \mathcal{H}_\alpha^0(\mathbb{T}^d)$, for μ-a.e. $\omega \in \Omega$, the sequence $(n^{-\frac{1}{2}}\|S_n(\omega, f)\|_2)$ has a limit $\sigma(f)$ which does not depend on ω.*

Moreover $\sigma(f) = 0$, if and only if f is a coboundary: there exists $h \in L^2(\nu)$ such that

$$f(t) = h(\theta\omega, \tau(\omega)t) - h(\omega, t), \ \nu - \text{a.e.} \tag{49}$$

Proof The convergence of the sequence of (global) variances (i.e. for the system $(\Omega \times X, \theta_\tau)$)

$$(n^{-1} \int\int |S_n(\omega, f)|^2 \, dt \, d\mu(\omega))_{n \geq 1}$$

to an asymptotic limit variance σ^2 is a general property of dynamical systems, for function with a summable decorrelation. In this case, we also know that $\sigma = 0$ if and only if f is a coboundary with a square integrable transfer function.

The system $(\Omega \times \mathbb{T}^d, \theta_\tau, \mu \times dt)$ is ergodic according to Proposition 3.3.

Denoting $F(\omega, t) := f(t)$, we have $S_n(\omega, f)(t) = \sum_{k=0}^{n-1} F(\theta_\tau^k(\omega, t))$, hence:

$$\begin{aligned}
\frac{1}{n}\|S_n(\omega, f)\|_2^2 &= \frac{1}{n}\sum_{\ell=0}^{n-1}\sum_{\ell'=0}^{n-1} \int_{\mathbb{T}^d} F(\theta_\tau^\ell(\omega, t))F(\theta_\tau^{l'}(\omega, t))\, dt \\
&= \|f\|^2 + \frac{2}{n}\sum_{r=1}^{n-1}\sum_{\ell=0}^{n-1-r} \int_{\mathbb{T}^d} F(\theta_\tau^\ell(\omega, t))F(\theta_\tau^{\ell+r}(\omega, t))\, dt \\
&= \|f\|^2 + 2\sum_{r=1}^{n-1} \frac{1}{n} \int_{\mathbb{T}^d} \sum_{\ell=0}^{n-1} (F.F \circ \theta_\tau^r)(\theta_\tau^\ell(\omega, t))\, dt \\
&- \ 2\sum_{r=1}^{n-1} \frac{1}{n} \int_{\mathbb{T}^d} F(\theta_\tau^\ell(\omega, t)) \sum_{\ell=n-r}^{n-1} F(\theta_\tau^{\ell+r}(\omega, t))\, dt.
\end{aligned}$$

Condition 3.2 insures, for a constant C and for a real $\kappa < 1$, the following inequality:

$$\begin{aligned}
&|\int_{\mathbb{T}^d} F(\theta_\tau^\ell(\omega, t))F(\theta_\tau^{\ell+r}(\omega, t))\, dt| \\
&= |\int_{\mathbb{T}^d} f(t)\ f(A(\theta^{\ell+r}\omega)...A(\theta^{\ell+1}\omega)t)\, dt| \leq C\|f\|_2\|f\|_\alpha \kappa^r.
\end{aligned} \tag{50}$$

This implies:

$$\left|\sum_{r=1}^{n-1} \int_{\mathbb{T}^d} F(\theta_\tau^\ell(\omega, t)) \frac{1}{n} \sum_{l=n-r}^{n-1} F(\theta_\tau^{\ell+r}(\omega, t))\, dt\right| \ \leq \ C\|f\|_2\|f\|_\alpha \frac{1}{n}\sum_{r=1}^{n-1} r\kappa^r,$$

hence this term tends to 0 if $n \to +\infty$ and the convergence of $\frac{1}{n}\|S_n(\omega, f)\|_2^2$ reduces to that of

$$\|f\|^2 + 2\sum_{r=1}^{n-1} \frac{1}{n} \int_{\mathbb{T}^d} \sum_{\ell=0}^{n-1} (F.F \circ \theta_\tau^r)(\theta_\tau^\ell(\omega, t))\, dt. \tag{51}$$

For μ-a.e. ω, for every r, by the ergodic theorem

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} \int_{\mathbb{T}^d} F(\theta_\tau^\ell(\omega,t))F(\theta_\tau^{\ell+r}(\omega,t))\,dt$$

$$= \lim_n \frac{1}{n} \sum_{\ell=0}^{n-1} \int_{\mathbb{T}^d} f(t)\ f(A(\theta^{\ell+r}\omega)...A(\theta^{\ell+1}\omega)t)\,dt = \int_{\Omega\times\mathbb{T}^d} (F.F \circ \theta_\tau^r)\,d\omega\,dt.$$

According to (50) we can take the limit for μ-a.e. ω in (51):

$$\lim_{n\to+\infty} \frac{1}{n} \|S_n(\omega,f)\|^2 = \|f\|_2^2 + 2\sum_{r=1}^{+\infty} \int_{\Omega\times\mathbb{T}^d} (F.F\circ\theta_\tau^r)\,d\omega\,dt = \lim_n \int\int \frac{1}{n}|S_n(\omega,f)|^2\,dt\ d\mu(\omega).$$

□

Remark 3.5. The previous proof shows that for a uniquely ergodic system (Ω,μ,θ) defined on a compact space Ω (for instance an ergodic rotation on a torus), the convergence of the variance given in Proposition 3.4 holds for every $\omega \in \Omega$, if the map τ is continuous outside a set of μ-measure 0.

3.2 Non-nullity of the variance

Now we consider more precisely the condition of coboundary. For $j,p \in \mathbb{Z}^d$, we denote by $D(j,p,\omega)$ the set $\{k \geq 0 : A_0^k(\omega)j = p\}$ and by $c(j,p,\omega) := \#D(j,p,\omega)$. (By convention, $A_0^0(\omega) = Id$.) We will use the following simple lemma:

Lemma 3.6. *Under Condition 3.2,* $\sup_{j\in J, p\in\mathbb{Z}^d}\ c(j,p,\omega) < \infty$*, for every finite subset* J *of* $\mathbb{Z}_*^d$.

Proof Let j be in J and let $k_1 := \inf\{k \in D(j,p,\omega)\}$. If k_2 belongs to $D(j,p,\omega)$ with $k_2 > k_1$, then $A_1^{k_2}(\omega)j = p = A_1^{k_1}(\omega)j$, so that: $A_{k_1+1}^{k_2}(\omega)j = j$. According to Condition 3.2, this implies that the number of such integers k_2 is finite and bounded independently of p. As J is finite, the result follows. □

Proposition 3.7. *Assume Condition 3.2. Let* f *be a trigonometric polynomial in* $L^2(\mathbb{T}^d)$*. If there exists* $g \in L^2(\Omega\times\mathbb{T}^d)$ *such that* $\int g\ d\nu = 0$ *and* $f = g - g\circ\theta_\tau$*, then* g *is also a trigonometric polynomial.*

Proof Let $f = \sum_{j\in J} f_j\chi_j$, where J is a finite subset of $\mathbb{Z}^d$. Let g be in L^2 such that $\int g\ d\nu = 0$ and $f(t) = g(\theta\omega, \tau(\omega)t) - g(\omega,t)$.

The coboundary relation implies $\sum_{k=0}^{N-1}(1-\frac{k}{N})f\circ\theta_\tau^k = g - \frac{1}{N}\sum_1^N g\circ\theta_\tau^k$. As g belongs to L^2, by ergodicity we deduce the convergence in L^2-norm

$$g = \lim_N \sum_{k=0}^{N-1}(1-\frac{k}{N})f\circ\theta_\tau^k,$$

with
$$\sum_{k=0}^{N-1}(1-\frac{k}{N})f\circ\theta_\tau^k = \sum_{p\in\mathbb{Z}^d}\sum_{k=0}^{N}[\sum_{j\,:\,A_0^k(\omega)j=p}(1-\frac{k}{N})\ f_j]\chi_p, \tag{52}$$

Moreover it is known that the maximal function $\sup_N \frac{1}{N}|\sum_1^N g\circ\theta_\tau^k|$ is square integrable. Therefore, by Fubini, for a.e. ω, there is $M(\omega)<\infty$ such that
$$\sup_N\sum_{p\in\mathbb{Z}^d}|\sum_{k=0}^{N}[\sum_{j\,:\,A_0^k(\omega)j=p}(1-\frac{k}{N})\ f_j]|^2 < M(\omega). \tag{53}$$

If N goes to ∞, the expression $\sum_{k=0}^N[\sum_{j\,:\,A_0^k(\omega)j=p}(1-\frac{k}{N})\ f_j]$ tends to the finite sum $\sum_{j\in J}c(j,p,\omega)\ f_j$ (cf. Lemma 3.6). According to (53), by restricting first the sums to a finite set of indices p and passing to the limit with respect to N in $\sum_p\ |\sum_{k=0}^N[\sum_{j\,:\,A_0^k(\omega)j=p}(1-\frac{k}{N})\ f_j]|^2$, we obtain finally
$$\sum_{p\in\mathbb{Z}^d}|\sum_{j\in J}c(j,p,\omega)\ f_j|^2 < M(\omega).$$

For every p, as J is finite and as $c(j,p,\omega)$ takes integral bounded values according to Lemma 3.6, $(|\sum_{j\in J}c(j,p,\omega)f_j|)_{p\in\mathbb{Z}^d}$ take only a finite number of distinct values. Let V be the set of these values and $\delta>0$ a lower bound of $V\setminus\{0\}$.

We have $\delta^2\ \#\{p\in\mathbb{Z}^d : \sum_{j\in J}c(j,p,\omega)f_j\neq 0\}\leq M(\omega)$, so that the cardinal is finite for a.e. ω. This shows that g is a trigonometric polynomial. $\square$

Corollary 3.8. *If f is a coboundary and has non negative Fourier coefficients, then $f(x)=0$ a.e.*

<u>Proof</u> By using the fact that $c(j,p,\omega)\in\mathbb{N}$, we get:
$$\begin{aligned}\|g(\omega,.)\|_2^2 &= \sum_p\left(\sum_{j\in J}c(j,p,\omega)f_j\right)^2 \geq \sum_p\left(\sum_{j\in J}c(j,p,\omega)f_j^2\right)\\ &\geq \sum_{j\in J}\left(\sum_p c(j,p,\omega)\right)f_j^2.\end{aligned}$$

For $j\neq 0$, we have $\sum_p c(j,p,\omega)=+\infty$, which implies $f_j=0$. $\square$

The previous results allow to obtain a "quenched" CLT (i.e. for a.e. ω) in the stationary case for positives matrices in $SL(2,\mathbb{Z})$, with (for trigonometric polynomials) a criterion of non-nullity of the variance. Moreover, when the Fourier coefficients of f are nonnegative, then the variance is >0.

3.3 $\mathcal{A} \subset SL(2, \mathbb{Z}^+)$

We consider in this subsection a finite set $\mathcal{A}$ of matrices in $SL(2, \mathbb{Z}^+)$ with positive coefficients. We study the asymptotical behavior of the products $A_i^j := A_i ... A_j$, where $A_i, ..., A_j$, $i \leq j$, is any choice of matrices in $\mathcal{A}$.

Let M be a 2×2 matrix with > 0 coefficients and having different real eigenvalues $r = r(M), s = s(M)$, $r > s$.

Let

$$\tilde{M} = \begin{pmatrix} r & 0 \\ 0 & s, \end{pmatrix} \; F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be respectively the diagonal matrix conjugate to M and the matrix such that $M = F\tilde{M}F^{-1}$, with $ad - bc = 1$.

Lemma 3.9. *The matrix M can be written:*

$$M = \begin{pmatrix} (r-s)u + s & -(r-s)v \\ (r-s)w & -(r-s)u + r \end{pmatrix},$$

with $u = ad \in]0, 1[$, $v = ab < 0$, $w = cd > 0$.

Proof The positivity of the coefficients of M implies that $v < 0$, $w > 0$. By multiplying the relation $ad - bc = 1$ by ad, we obtain $u^2 - vw = u$, thus $u^2 - u = vw < 0$. $\square$

Lemma 3.10. *There exist a constant C such that for every p in $\mathbb{Z}^2_*$, and every product M of n matrices taking values in $\mathcal{A}$, if $n \geq C \ln \|p\|$, then $Mp \in \mathbb{R}^2_+ \cup \mathbb{R}^2_-$.*

Proof Let $\lambda := \frac{w}{u} = \frac{u-1}{v}$. We have $\lambda > 0$ and we can rewrite the matrix M as

$$M = r \begin{pmatrix} u & \lambda^{-1}(1-u) \\ \lambda u & 1-u \end{pmatrix} + s \begin{pmatrix} 1-u & -\lambda^{-1}(1-u) \\ -\lambda u & u \end{pmatrix}.$$

Thus, for every vector $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $MX = r(ux + \lambda^{-1}(1-u)y) \begin{pmatrix} 1 \\ \lambda \end{pmatrix} + s(x - \lambda^{-1}y) \begin{pmatrix} 1-u \\ -\lambda u \end{pmatrix}$.

The eigenvectors of M are $\begin{pmatrix} 1 \\ \lambda \end{pmatrix}$ and $\begin{pmatrix} 1-u \\ -\lambda u \end{pmatrix}$, corresponding respectively to the eigenvalues r and s.

As M is a product of n matrices of $\mathcal{A}$, it maps the cone $\mathbb{R}^2_+$ strictly into itself:

$$M\mathbb{R}^2_+ \subset \bigcup_{A \in \mathcal{A}} A\mathbb{R}^2_+.$$

It follows that the slope $\lambda = \lambda(M)$ of the positive eigenvector of M is bounded from below and above by constants which only depend on $\mathcal{A}$: there exists $\delta > 0$ such that $\delta \leq \lambda \leq \delta^{-1}$.

Let us write $r\zeta + \varphi$ and $\lambda r \zeta + \psi$ the components of MX with:

$$\zeta := ux + \lambda^{-1}(1-u)y, \; \varphi := s(x - \lambda^{-1}y)(1-u), \; \psi := -s(x - \lambda^{-1}y)\lambda u.$$

There exist constants $C' > 0$ and $\gamma > 1$ such that the positive eigenvalue $r(M)$, for M a product of n matrices taking values in $\mathcal{A}$, satisfies: $r(M) \geq C'\gamma^n$.

As $s(M) = r(M)^{-1}$, we have $s(M) \leq C'^{-1}\gamma^{-n}$ and, as $\delta \leq \lambda \leq \delta^{-1}$,

$$\max(|\varphi|, |\psi|) \leq C'^{-1}\delta^{-1}\gamma^{-n}\|X\|.$$

Let $X \in \mathbb{Z}^2$ be non zero. Up to a replacement of X by $-X$, we can assume that $\zeta \geq 0$. The vector MX having non zero integer coordinates, we have:

$$r\zeta + |\varphi| + \lambda r\zeta + |\psi| \geq |r\zeta + \varphi| + |\lambda r\zeta + \psi| \geq 1.$$

Thus:

$$r\zeta \geq \frac{1}{1+\lambda} - \frac{1}{1+\lambda}(|\varphi| + |\psi|),$$

and

$$r\zeta + \varphi \geq \frac{1}{1+\lambda} - \frac{1}{1+\lambda}((2+\lambda)|\varphi| + |\psi|), \qquad \lambda r\zeta + \psi \geq \frac{\lambda}{1+\lambda} - \frac{1}{1+\lambda}(\lambda|\varphi| + (1+2\lambda)|\psi|).$$

As $\max(|\varphi|, |\psi|) \leq C'^{-1}\delta^{-1}\gamma^{-n}\|X\|$, there exists $C > 0$ such that if $n \geq C \ln \|p\|$ then $r\zeta + \varphi > 0$ and $\lambda r\zeta + \psi > 0$ that is $MX \in \mathbb{R}^2_{+*}$. $\square$

Corollary 3.11. *Let $(A_k)_{k\geq 1}$ be a sequence of matrices taking values in $\mathcal{A}$. Denote by $\tau_k : x \to A_k x \bmod \mathbb{Z}^2$ the corresponding automorphisms of the torus. Then, for almost every x in $\mathbb{T}^2$, the sequence $(\tau_k ... \tau_1 x)_{k\geq 1}$ is equidistributed in $\mathbb{T}^2$.*

Corollary 3.12. *There exist constants $C_1 > 0, \gamma > 1$, and c such that for every $p \in \mathbb{Z}^2 \setminus \{0\}$:*

$$\|A_1^{\ell+r} p\| \ \geq C_1 \gamma^{r - c\log\|p\|} \|A_1^\ell\|, \ \forall \ell, r \geq 1. \tag{54}$$

For vectors $q \in \mathbb{Z}^2_+$ belonging to some cone strictly contained in the positive cone, the norm $\|A^n q\|$ is comparable to the norm $\|A^n\|$ and there are constants $C > 0$ and $\lambda > 1$ such that $\|A^n q\| \geq C\lambda^n$. Therefore, $\mathcal{S}(D, \Delta)$ is a consequence of (54).

Corollary 3.13. *For every $D > 0$ there exists Δ such that $\mathcal{S}(D, \Delta)$ holds with respect to any products of matrices in $\mathcal{A}$.*

Proof Let us suppose that $\sum_{j=1}^s [A_1^{\ell'_j} p'_j + A_1^{\ell_j} p_j] = 0$, i.e.

$$A_1^{\ell'_s} p'_s + A_1^{\ell_s} p_s = -\sum_{j=1}^{s-1} [A_1^{\ell'_j} p'_j + A_1^{\ell_j} p_j]. \tag{55}$$

Inequality (54) ensures inequalities such as:

$$\|A_1^{\ell_j} p_j\| \leq D\|A_1^{\ell_j}\| \leq C_1^{-1} D \gamma^{-(\ell_s - \ell_j) + c\ln\|q_s\|} \|A_1^{\ell_s} q_s\|,$$

for $q_s \in \mathbb{Z}^2_*$ and $\|p_j\| \leq D$. Then, using the gaps between the ℓ_j we will get a contradiction. More precisely we consider two cases.

1) $\ell'_s - \ell_s$ small: $0 \leq \ell'_s - \ell_s \leq \rho_1$, where ρ_1 will be defined later.

Write $q_s = A_{\ell_s+1}^{\ell'_s} p'_s + p_s$. This is a non-zero vector in $\mathbb{Z}^2$ and its norm is less than $2D \max_{A\in\mathcal{A}} \|A\|^{\rho_1}$. Let $C_2 := \ln \max_{A\in\mathcal{A}} \|A\|$. We deduce from (54):

$$\begin{aligned}
\|A_1^{\ell'_s} p'_s + A_1^{\ell_s} p_s\| &= \|A_1^{\ell_s} q_s\| \\
&\leq C_1^{-1} D[\sum_{j=1}^{s-1} \gamma^{-(\ell_s-\ell'_j)+c\ln\|q_s\|}\|A_1^{\ell_s} q_s\| + \sum_{j=1}^{s-1} \gamma^{-(\ell_s-\ell_j)+c\ln\|q_s\|}\|A_1^{\ell_s} q_s\|] \\
&\leq C_1^{-1} D\gamma^{c\ln\|q_s\|}\|A_1^{\ell_s} q_s\| \, [\sum_{j=1}^{s-1} \gamma^{-(\ell_s-\ell'_j)} + \sum_{j=1}^{s-1} \gamma^{-(\ell_s-\ell_j)}] \\
&\leq 4C_1^{-1} D'\gamma^{cC_2\rho_1} \, [\sum_{j=1}^{s-1} \gamma^{-j\Delta)}] \, \|A_1^{\ell_s} q_s\| \\
&\leq \frac{4}{C_1(1-\gamma^{-\Delta})} D'\gamma^{cC_2\rho_1-\Delta} \, \|A_1^{\ell_s} q_s\|,
\end{aligned}$$

with $D' = \gamma^{cln(2D)} D$.

2) $\ell'_s - \ell_s \geq \rho_1$.

We can assume that $p'_s \neq 0$. Otherwise we would have $p_s \neq 0$ and we would consider $\|A_1^{\ell_s} p_s\|$ instead of $\|A_1^{\ell'_s} p'_s\|$. Still using (54) we get:

$$\begin{aligned}
\|A_1^{\ell'_s} p'_s\| &\leq \|A_1^{\ell_s} p_s\| + C_1^{-1} D[\sum_{j=1}^{s-1} \gamma^{-(\ell'_s-\ell'_j)+c\ln\|p'_s\|}\|A_1^{\ell'_s} p'_s\| + \sum_{j=1}^{s-1} \gamma^{-(\ell'_s-\ell_j)+c\ln\|p'_s\|}\|A_1^{\ell'_s} p'_s\|] \\
&\leq C_1^{-1} D\gamma^{c\ln\|p'_s\|}\|A_1^{\ell'_s} p'_s\| \, [\gamma^{-(\ell'_s-\ell_s)} + \sum_{j=1}^{s-1} \gamma^{-(\ell'_s-\ell'_j)} + \sum_{j=1}^{s-1} \gamma^{-(\ell'_s-\ell_j)}] \\
&\leq C_1^{-1} D[\gamma^{c\ln D-\rho_1} + 2\frac{\gamma^{c\ln D-\Delta}}{(1-\gamma^{-\Delta})}] \, \|A_1^{\ell'_s} p'_s\|.
\end{aligned}$$

Chose ρ_1 such that $C_1^{-1} D\gamma^{c\ln D-\rho_1} < \frac{1}{2}$, then Δ such that

$$2C_1^{-1}\frac{D\gamma^{c\ln D-\Delta}}{(1-\gamma^{-\Delta})} < \frac{1}{2},$$
$$\frac{4}{C_1(1-\gamma^{-\Delta})} D'\gamma^{cC_2\rho_1-\Delta} < 1.$$

The factor in front of $\|A_1^{\ell_s} q_s\|$ on the right in the first case is < 1 and the factor in front of $\|A_1^{\ell'_s} p'_s\|$ on the right in the second case is < 1. In both cases there is a contradiction. $\square$

Corollary 3.13 and Inequality (7) enable us to prove a CLT for the action of sequences A_1^n. Let $g_n = g$ be a fixed trigonometric polynomial such that $\hat{g}(p) = 0$ if $\|p\| > D$. Let us take Δ such that $\mathcal{S}(D, \Delta)$ holds (via Corollary 3.13) and remark that in the case of $SL(2, \mathbb{Z}^+)$ that we are studying the numbers $\sigma_{k,n}^{\frac{1}{2}}$ are bounded by $Cn^{\beta/2}$. Inequality (7) of Lemma 1.4 becomes:

$$\begin{aligned} & |\mathbb{E}[e^{ix\frac{S_n}{\|S_n\|_2}}] - e^{-\frac{1}{2}x^2}| \\ \leq \ & C[|x|\|S_n\|_2^{-1} n^{\frac{1-\beta}{2}} + |x|^3 \|S_n\|_2^{-3} n^{1+2\beta} + |x|\|S_n\|_2^{-1} n^{\frac{1+3\beta}{4}} \\ & + |x|^2 \|S_n\|_2^{-1} n^{\frac{1-\beta}{2}} \Delta + |x|^2 \|S_n\|_2^{-2} n^{1-\beta} \Delta^2]. \end{aligned} \tag{56}$$

If we suppose that $\|S_n\|_2 \geq Cn^{\delta}$, we get:

$$\begin{aligned} & |\mathbb{E}[e^{ix\frac{S_n}{\|S_n\|_2}}] - e^{-\frac{1}{2}x^2}| \\ \leq \ & C[|x| n^{-\frac{(\beta-1+2\delta)}{2}} + |x|^3 n^{-(-2\beta-1+3\delta)} + |x| n^{-\frac{(-3\beta-1+4\delta)}{4}} \\ & + |x|^2 n^{-\frac{(\beta-1+2\delta)}{2}} \Delta + |x|^2 n^{-(\beta-1+2\delta)} \Delta^2]. \end{aligned} \tag{57}$$

Inequality of Esseen

If X, Y are two r.r.v.'s defined on the same probability space, their mutual distance in distribution is defined by:

$$d(X, Y) = \sup_{x \in \mathbb{R}} |\mathbb{P}(X \leq x) - \mathbb{P}(Y \leq x)|.$$

Let be $H_{X,Y}(x) := |\mathbb{E}(e^{ixX}) - \mathbb{E}(e^{ixY})|$. Take as Y a r.v. Y_σ with a normal law $\mathcal{N}(0, \sigma^2)$.

Recall the following inequality (cf. Feller, *An introduction to probability theory and its application*, p. 512): if X has a vanishing expectation and if the difference of the distributions of X and Y vanishes at $\pm\infty$, then for every $U > 0$,

$$d(X, Y_\sigma) \leq \frac{1}{\pi} \int_{-U}^{U} H_{X,Y}(x) \frac{dx}{x} + \frac{24}{\pi} \frac{1}{\sigma\sqrt{2\pi}} \frac{1}{U}.$$

Taking $X = S_n / \|S_n\|_2$, we have here that $|H_{X,Y_{\sigma_n}}| \leq \sum_{i=1}^{5} n^{-\gamma_i} |x|^{\alpha_i}$, where the constants are given by (57). Thus $d(X, Y_1)$ is bounded by

$$\frac{C}{U} + \sum_{i=1}^{5} n^{-\gamma_i} \frac{1}{\alpha_i} U^{\alpha_i}.$$

In order to optimize the choice of $U = U_n$, we take $U_n = n^{\gamma}$ with $\gamma = \min_i \frac{\gamma_i}{\alpha_i + 1}$. This gives the bound

$$d(\frac{S_n}{\|S_n\|_2}, Y_1) \leq Cn^{-\gamma}.$$

We have to compute

$$\begin{aligned}\gamma &= \min(\frac{\beta-1+2\delta}{4}, \frac{-2\beta-1+3\delta}{4}, \frac{-3\beta-1+4\delta}{8}, \frac{\beta-1+2\delta}{6}, \frac{\beta-1+2\delta}{3}) \\ &= \min(\frac{-2\beta-1+3\delta}{4}, \frac{-3\beta-1+4\delta}{8}, \frac{\beta-1+2\delta}{6}). \end{aligned} \tag{58}$$

For $\delta = \frac{1}{2}$ we get: $\gamma = \min(\frac{-4\beta+1}{8}, \frac{-3\beta+1}{8}, \frac{\beta}{6}) = \min(\frac{-4\beta+1}{8}, \frac{\beta}{6})$. Taking $\beta = \frac{3}{16}$, we obtain $\gamma = \frac{1}{32}$. This gives a rate of convergence of order $n^{-\frac{1}{32}}$.

Theorem 3.14. *Let $(A_k)_{k\geq 1}$ be a sequence of matrices taking values in a finite set $\mathcal{A}$ of matrices in $SL(2,\mathbb{Z}_+)$ with > 0 coefficients. If, for a constant $C_1 > 0$ and a rank n_0, $\|S_n\| \geq C_1 n^{\frac{1}{2}}$, for $n \geq n_0$, then for a constant C we have:*

$$d(\frac{S_n}{\|S_n\|_2}, Y_1) \leq C n^{-\frac{1}{32}}, \forall n \geq n_0. \tag{59}$$

The previous results can be applied if the limit of $n^{-\frac{1}{2}} \|S_n\|_2$ exists and is non zero: the sequence $(n^{-\frac{1}{2}} S_n)_{n\geq 1}$ then tends in distribution towards the normal law $N(0,1)$ with a rate given by (59).

We can also obtain a rate of convergence of order $n^{-\delta}$, for some $\delta > 0$, for subsequences provided that the variance $\|S_{n_k}\|_2$ is large enough:

Along a subsequence (n_k) such that $\|S_{n_k}\|_2 \geq C_1 n_k^\delta$, with $\delta > 3/7$, the subsequence of normalized sums $(\|S_{n_k}\|_2^{-1} S_{n_k})$ converges in distribution towards the normal law $\mathcal{N}(0,1)$.

Indeed, in (58), to obtain a strictly positive γ, we have to check the inequalities:

$$-2\beta - 1 + 3\delta > 0,\ -3\beta - 1 + 4\delta > 0,\ \beta - 1 + 2\delta > 0.$$

that is:

$$1 - 2\delta < \beta < \min(\frac{3\delta-1}{2}, \frac{4\delta-1}{3}) = \frac{3\delta-1}{2}.$$

For $\delta > \frac{3}{7}$ and $\beta = \frac{1}{7}$, we have $\gamma > 0$.

Remarks 3.15. 1) In the previous statements, we have considered the case of trigonometric polynomials. Using some approximation, it can be extended to Hölder continuous functions or characteristic functions of a regular set.

2) If the sequence (A_n) is generated by a dynamical system (Ω, θ, μ), we have shown that in the case of $SL(2,\mathbb{Z}^+)$-matrices, that either for μ-almost $\omega \in \Omega$, $(\|S_n(\omega, f)\|_2)$ is bounded or, μ-almost $\omega \in \Omega$, the sequence $(n^{-\frac{1}{2}}\|S_n(\omega, f)\|_2)$ has a limit $\sigma(f) > 0$ not depending on ω. In the later case, the CLT holds.

For instance (cf. Remark 3.5), if the sequence (A_n) is generated by an ergodic rotation on the circle, with $A(\omega) = A$ on an interval and $= B$ on the complementary, then we obtain the CLT for every such sequence.

3) If the dynamical system (Ω, θ, μ) is weakly mixing, then the characteristic function of a regular set is never a coboundary for the extended system. Thus we necessarily have $\sigma(f) > 0$. That is to say that, if (Ω, θ, μ) is weakly mixing, the CLT holds almost surely for centered characteristic functions of regular sets.

4 Appendix

Proof of Lemma 1.1 1) Setting $\psi(y) = (1+iy)e^{-\frac{1}{2}y^2}e^{-iy}$ and writing $\psi(y) = \rho(y)e^{i\theta(y)}$, where $\rho(y) = |\psi(y)|$, we have

$$\ln \rho(y) = \frac{1}{2}[\ln(1+y^2) - y^2] \leq 0, \ \tan(\theta(y)) = \frac{y - \tan y}{1 + y \tan y}.$$

An elementary computation gives the following upper bounds for some constant C_1:

$$|\ln \rho(y)| \leq \frac{1}{4}|y|^4, \ |\theta(y)| \leq C_1|y|^3, \ \forall y \in [-1, 1]. \tag{60}$$

Let us write: $Z(x) = Q(x) \exp(-\frac{1}{2}x^2 \ Y) \ [\prod_{k=0}^{u-1} \psi(x\zeta_k)]^{-1}$. Using the fact that $\ln \rho(x\zeta_k) \leq 0$, we have:

$$\begin{aligned} |Z(x) - Q(x) \exp(-\frac{1}{2}x^2 \ Y)| &= |Z(x) - Z(x) \prod_{k=0}^{u-1} \psi(x\zeta_k)| = |1 - \prod_{k=0}^{u-1} \psi(x\zeta_k)| \\ &\leq |1 - e^{\sum_{k=0}^{u-1} \ln \rho(x\zeta_k)}| + |1 - e^{i\sum_{k=0}^{u-1} \theta(x\zeta_k)}| \\ &\leq \sum_{k=0}^{u-1} |\ln \rho(x\zeta_k)| + \sum_{k=0}^{u-1} |\theta(x\zeta_k)|. \end{aligned}$$

If $|x|\delta \leq 1$, where $\delta = \max_k \|\zeta_k\|_\infty$, we can apply the bound (60). Using the inequality

$$|1 - e^s| \leq (e-1)|s| \leq 2|s|, \forall s \in [-1, 1] \tag{61}$$

we obtain for a constant C:

$$|Z(x) - Q(x) \exp(-\frac{1}{2}x^2 \ Y)| \leq C|x|^3 \sum_{k=0}^{u-1} |\zeta_k|^3 \leq Cu|x|^3\delta^3.$$

2) Since Y is a positive random variable, we have also:

$$\left|\exp(-\frac{1}{2}x^2 Y) - \exp(-\frac{1}{2}a\, x^2)\right| \leq \frac{x^2}{2} \, |Y - a| \, .$$

If $|x|\delta \leq 1$ we get:

$$\begin{aligned} & |Z(x) - \exp(-a\,\frac{x^2}{2})Q(x)| \\ \leq\ & |Z(x) - Q(x)\exp(-\frac{x^2}{2}Y)| + |Q(x)\,[\exp(-\frac{x^2}{2}Y) - \exp(-a\,\frac{x^2}{2})]| \\ \leq\ & C u\,|x|^3\delta^3 + \frac{x^2}{2}|Q(x)|\,|Y-a|; \end{aligned}$$

hence, under the condition $|x|\delta \leq 1$, we obtain the upper bound (2):

$$\begin{aligned} |\mathbb{E}[Z(x)] - \exp(-a\,\frac{x^2}{2})| &= |\mathbb{E}[Z(x) - e^{-\frac{1}{2}a\,x^2}Q(x) + e^{-\frac{1}{2}a\,x^2}(Q(x)-1)]| \\ &\leq |\mathbb{E}[Z(x) - e^{-\frac{1}{2}a\,x^2}Q(x)]| + e^{-\frac{1}{2}a\,x^2}|\mathbb{E}[Q(x)-1)]| \\ &\leq C u\,|x|^3\delta^3 + \frac{x^2}{2}\mathbb{E}[|Q(x)||Y-a|] + e^{-\frac{1}{2}a\,x^2}|1 - \mathbb{E}[Q(x)]| \\ &\leq C u\,|x|^3\delta^3 + \frac{x^2}{2}\|Q(x)\|_2\|Y-a\|_2 + |1 - \mathbb{E}[Q(x)]|. \end{aligned}$$

3) The bound that we obtain is large in general, because the integral of $Q(x)$ is of order $e^{\frac{1}{2}\sigma^2 x^2}$ and the bound for $Q(x)$ is very large if x is big. If $e^{-\frac{1}{2}a\,x^2}\,\|Q(x)\|$ is bounded, we can obtain a more accurate upper bound.

For $0 \leq \varepsilon \leq 1$, let $A_\varepsilon(x) = \{\omega : x^2|Y(\omega) - a| \leq \varepsilon\}$. We have the following bounds:

$$\begin{aligned} & |\mathbb{E}[1_{A_\varepsilon(x)}(Z(x) - \exp(-a\,\frac{x^2}{2}))]| \\ \leq\ & C\,u\,|x|^3\delta^3 + \mathbb{E}[1_{A_\varepsilon(x)}\,(|Q(x)\,[\exp(-\frac{x^2}{2}Y) - e^{-\frac{1}{2}a\,x^2}]|)] \\ +\ & e^{-\frac{1}{2}a\,x^2}[|1 - \mathbb{E}(Q(x)| + \mathbb{E}(1_{A^c_\varepsilon(x)}|1 - Q(x)|)] \\ \leq\ & C\,u\,|x|^3\delta^3 + e^{-\frac{1}{2}a\,x^2}\|Q(x)\|_2\|1_{A_\varepsilon(x)}[\exp(-\frac{x^2}{2}(Y-a)) - 1]\|_2 \\ +\ & e^{-\frac{1}{2}a\,x^2}[|1 - \mathbb{E}(Q(x))| + \mathbb{E}(1_{A^c_\varepsilon(x)}|1 - Q(x)|)]. \end{aligned}$$

From (61) we have

$$\|1_{A_\varepsilon(x)}\,[\exp(-\frac{x^2}{2}(Y-a)) - 1]\|_2 \leq 2\varepsilon,$$

and using Cauchy-Schwarz inequality, we get

$$\mathbb{E}(1_{A^c_\varepsilon(x)}|1 - Q(x)|)] \leq \mathbb{P}(A^c_\varepsilon(x)) + \|Q(x)\|_2(\mathbb{P}(A^c_\varepsilon(x)))^{\frac{1}{2}},$$

which implies:

$$\begin{aligned} & |\mathbb{E}[1_{A_\varepsilon(x)}(Z(x) - \exp(-a\,\frac{x^2}{2}))]| \\ \leq\ & C\,u\,|x|^3\delta^3 + e^{-\frac{1}{2}a\,x^2}\|Q(x)\|_2\,\varepsilon \\ +\ & e^{-\frac{1}{2}a\,x^2}[|1 - \mathbb{E}(Q(x))| + \mathbb{P}(A^c_\varepsilon(x)) + \|Q(x)\|_2(\mathbb{P}(A^c_\varepsilon(x)))^{\frac{1}{2}}] \\ \leq\ & C\,u\,|x|^3\delta^3 + e^{-\frac{1}{2}a\,x^2}\|Q(x)\|_2\,[\varepsilon + (\mathbb{P}(A^c_\varepsilon(x)))^{\frac{1}{2}}] + e^{-\frac{1}{2}a\,x^2}[|1 - \mathbb{E}(Q(x))| + \mathbb{P}(A^c_\varepsilon(x))]. \end{aligned}$$

Choosing $\varepsilon = |x| \|Y-a\|_2^{\frac{1}{2}}$, we get

$$\mathbb{P}(A_\varepsilon^c(x)) \leq \varepsilon^{-2} x^4 \|Y-a\|_2^2 \leq x^2 \|Y-a\|_2.$$

This yields:

$$\begin{aligned} & |\mathbb{E}[1_{A_\varepsilon(x)}(Z(x) - \exp(-a\,\frac{x^2}{2})]| \\ \leq \quad & C\,u\,|x|^3\delta^3 + 2|x|\,e^{-\frac{1}{2}a\,x^2}\,\|Q(x)\|_2\,\|Y-a\|_2^{\frac{1}{2}} + e^{-\frac{1}{2}a\,x^2}\,[|1-\mathbb{E}(Q(x))| + x^2\|Y-a\|_2]. \end{aligned}$$

Thus, assuming $|x|\delta \leq 1$ and $|x| \|Y-a\|_2^{\frac{1}{2}} \leq 1$, we obtain (3):

$$\begin{aligned} & |\mathbb{E}[Z(x)] - \exp(-\frac{1}{2}a\,x^2)| \leq |\mathbb{E}[1_{A_\varepsilon(x)}(Z(x) - \exp(-a\,\frac{x^2}{2}))]| + 2\mathbb{P}(A_\varepsilon^c(x)) \\ \leq \quad & C\,u\,|x|^3\delta^3 + 2\,|x|\,e^{-\frac{1}{2}a\,x^2}\,\|Q(x)\|_2\,\|Y-a\|_2^{\frac{1}{2}} + e^{-\frac{1}{2}a\,x^2}\,[|1-\mathbb{E}(Q(x))|] + 3\,|x|^2\|Y-a\|_2. \end{aligned}$$

□

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